DIGITAL SIGNAL PROCESSING

BEG 433 EC

.Year: IV Semester: II

Teaching Schedule Hours/Week			Examination Scheme				
Theory	Tutorial	Practical	Internal Assessment		Final		Total
3	-	3/2	Theory	Practical*	Theory	Practical**	125
			20	25	80	-	

*** Continuous**

**** Duration:** 3 hours

Course objectives: To provide

1. Discrete signals **5**

1.1 Discrete signals – unit impulse, unit step, exponential sequences
1.2 Linearity, shift invariance, causality
1.3 Convolution summation and discrete systems, response to discrete inputs
1.4 Stability sum and convergence of power series
1.5 Sampling continuous signals spectral properties of sampled signals

2. The discrete Fourier transforms **5**

2.1 The discrete Fourier transform (DFT) derivation
2.2 Properties of the DFT, DFT of non-periodic data
2.3 Introduction of the fast fourier transform (FFT)
2.4 Power spectral density using DFT/FFT algorithms

3. Z transform **8**

3.1 Definition of Z transform one sided and two sided transforms
3.2 Region of convergence relationship to causality
3.3 Inverse Z transform – by long division, by partial fraction expansion.
3.4 Z transform properties – delay advance, convolution, Parseval's theorem
3.5 Z transforn transfer function H (Z) –transient and steady state sinusoidal response pole zero relationships, stability
3.6 General form of the linear, shift invariant constant coefficient difference equation
3.7 Z transform of difference equation.

4. Frequency response **4**

4.1 Steady state sinusoidal frequency response derived directly from the difference equation
4.2 Pole zero diagrams and frequency response
4.3 Design of a notch filter from the pole zero diagram.

5. Discrete filters **6**

5.1 Discrete filters structures, second order sections ladder filters frequency response
5.2 Digital filters finite precision implementations of discrete filters
5.3 Scaling and noise in digital filters, finite quantized signals quantization error linear models.

6. HR Filter Design **7**

6.1 Classical filter design using polynomial approximations – Butterworth Chebishev

6.2 HR filter design by transformation matched Z transform impulse, invariant transform and bilinear transformation

6.3 Application of the bilinear transformation to HR low pass discrete filter design

6.4 Spectral transformations, high pass, band pass and notch filters.

7. FIR Filter Design **3**

7.1 FIR filter design by fourier approximation the complex fourier series

7.2 Gibbs phenomena in FIR filter design approximations, applications of window Functions to frequency response smoothing rectangular hanning Hamming and Kaiser windows.

7.3 FIR filter design by the frequency sampling method

7.4 FIR filter design using the Remez exchange algorithm

8. Digital filter Implementation

8.1 Implementations using special purpose DSP processors, the Texas Instruments TMS320.

8.2 Bit serial arithmetic distributed arithmetic implementations, pipelined implementations

Laboratory:

1. Introduction to digital signals sampling properties, aliasing, simple digital notch filter behaviour
2. Response of a recursive (HR) digital filter comparison to ideal unit sample and frequency response coefficient quantization effects.
3. Scaling dynamic range and noise behaviour of a recursive digital filter, observation of nonlinear finite precision effects.

Digital Signal processing:-

Application:-

1) Speech recognition.
2) Telecommunication.

Digital Signal processing over analog signal processing:

1) Accuracy.
2) Offline processing.
3) Software control.
4) Cheaper than analog counterpart.

Basic element of D.S.P system:-

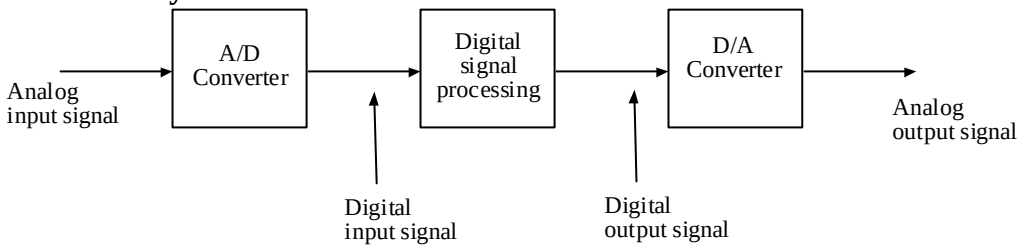


Fig: Block diagram of DSP

- Most of the signal encountering science and engineering are analog in nature i.e the signals are function of continuous variable substance in usually take on value in a continuous range.
- To perform the signal processing digitally, there is need for interface between the analog signal and digital processor. This interface is called analog to digital converter. The o/p of A/D converter is digital signal i.e appropriate as an i/p to the digital processor.
- Digital signal processor may be a large programmable digital computer or small microprocessor program to perform the desired operation on i/p signal.
- It may a also be a hardwired digital processor configure to perform a specified set of operation on the i/p signal.
- Programming machine provide the flexibility to change the signal processing operation through a change in software whereas hardwired m/c are difficult to reconfigure.
- In application where the distance o/p from digital signal processor is to be given to the user in analog form, we must provide another interface on the digital domain into analog domain. Such an interface is called D/A converter.

Advantage of Digital over analog signal processing:-

1. A Digital programmable system allows flexibility in reconfiguring the digital signal processing operation simply by changing the program. Reconfiguration of analog system usually implies redesign of hardware followed by testing and verification to see that if operates properly.
2. Digital system provide much better control of accuracy requirements.
3. Digital system are easily stored on magnetic media without loss of signal …… beyond that introduce in A/D conversion. As a consequence, the signals become transportable and can be processed offline in a remote laboratory.

4. Digital signal processing method also allows for the implementation of more sophisticated.
5. In some cases a digital implementation of signal processing system is cheaper than its analog counter part.

*** Signal:**

It is defined as any physical quantity which is a function of one or more independent variable and contains some information.

In electrical sense, the signal can be voltage or current. The voltage or current is a function of time as an independent variable.

The independent variable in the mathematical representation of a signal may be either continues or discrete.

Continues time signals are defined analog continues times. Contineous time signals are often referred to as analog signals.

Discrete time signals are defined as certain time instant.

Digital signals are those for which both time and amplitude discrete.

Discrete- time signals or sequence:-

D.T.S are re……. mathematically as a sequence of numbers. A sequence of numbers x in which the n^{th} no in the sequence is denoted by x(n) and written as:

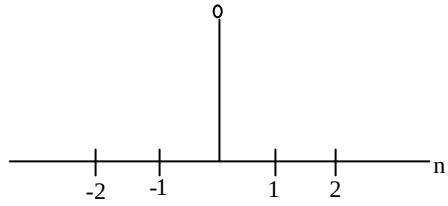
x = { x(n)} - infinity < n < infinity

Figure:

- **unit sample or unit impulse sequence:**

 It is denoted by δ(n) and defined as:

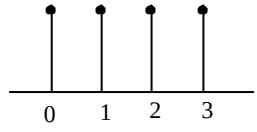
$$\delta(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 1 \end{cases}$$



*** Unit step sequence:**

It is denoted by U(n) and is defined as:

U(n) = 1 , n ≥ 0
= 0, n<0



U(n) = δ(n) + δ(n-1) + δ(n-2)+ ……

$$= \sum_{0}^{\infty} \delta(n-k)$$

$$U(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

i.e the value of unit step sequence at time n is equal to the accumulated sum of value at index n and all prvious value of impulse sequence.

Conversely the impulse sequence can be expressed as the first backward difference of unit step sequences.

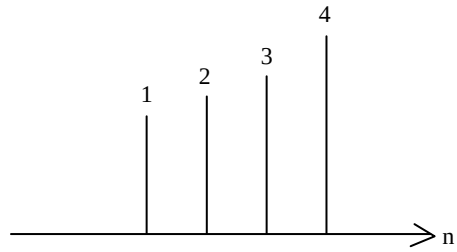
i.e. $\delta(n) = u(n) - u(n-1)$

*** Unit ramp sequence:-**

It is denoted by $u_r(n)$ and defined as

$u_r(n) = n \quad n \geq 0$

$= 0 \quad n < 0$



*** Exponential Sequence:-**

The exponential signal is a sequence of the form $x(n) = a^n$ for all n.

If the parameter 'a' is real, then x(n) is real signal. Fig illustrate x(n) for various values of parameter 'a'.

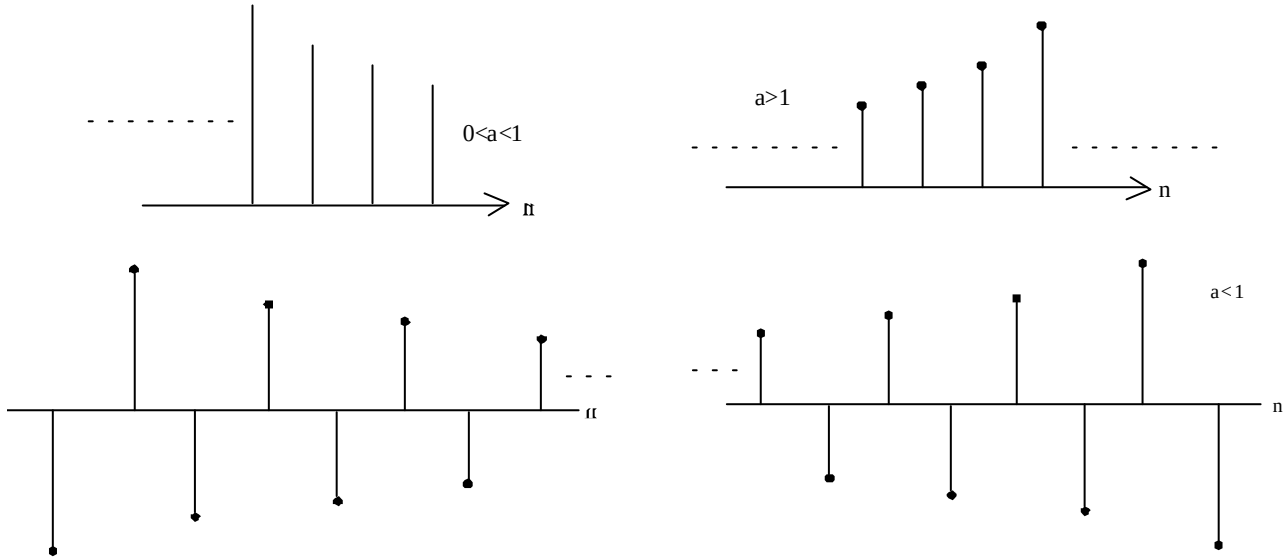


Fig: Graphical representation of Exponential signals.

$0 < a < 1$:

Eg $a = ½$

i.e $(1/2)^n = 0, ½, ¼, 1/8$ (exponential decreasing)

(see figure (i))

a > 1,
E.g , a = 2
i.e $(2)^n$ = 1,2, 4, 8 (Exponential increasing)
(See figure (ii))

-1 < a <0
Eg.: -1/2
i.e $(-1/2)^n$ = 1, -1/2, ¼, -1/8
(See fig (iii))

a <-1
E.g a = -2
$(-2)^n$ = 1, -2, 4, -8
(See figure(iv))

Exponential sequence:-

When the parameter 'a' is complex values , it can be expressed as:

$a = re^{j\theta}$

Where r and θ are new parameters. Hence, we can express x(n) as:

$X(n) = r^n e^{j\theta}$

$= r^n (\cos \theta n + j \sin \theta n)$

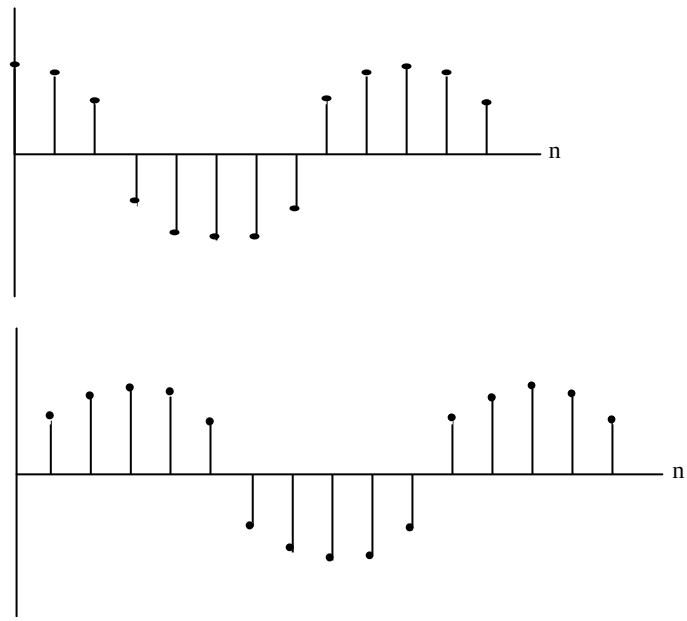
Since, x(n) is now complex values, it can represented graphically by plotting the real part,

$x_e(n) = r^n \cos \theta n$ as a function of n and separately plotting the imaginary part.

$x_i(n) = r \sin \theta n$ as a function of n.

Fig. illustrates the graphs fo

$X_e(n)$ and $x_i(n)$.



We observe that the signals $x_e(n)$ and $x_i(n)$ are damaged (decaying exponentially, i.e r <r) cosine function and damped sine function.

If r =1 , the damping disappears and $x_e(n)$, $x_i(n)$ and x(n) have fixed amplitude which is unity.

Alternatively, the signal x(n) can be represented graphically by the amplitude function.

$|x(n)| = A(n) = r^n$

And phase function

$\angle x(n) = \phi(n) = \theta_n$

Representation of discrete-time signal: -

1) Functional representation:-

E.g x(n) = 1 , for n = 1,3
= 4, for n = 2
= 0 elsewhere,

2) Tabular representation:

n	----- 2	-1	0	1	2	3	4
x(n)	----- 0	0	0	1	4	1	0

3) Sequence representation:

X(n) = { ……0 , 0, 1, 4, 1, 0, …….} Infinite duration .
X(n) = { 0, 1, 4, 1) finite duration (4- point sequence)

4) Graphical representation:-
Figure:

Date: 2066/05/23

Linearity: A system is called liner of superposition principal applies to that system. This means that the liner system may be defined as one whose response to the sum of weighted inputs is same as the sum of weighted response.

Let us consider a system. If $x_1(n)$ is the input and $y_1(n)$ is the output. Similarly $y_2(n)$ is the response to $x_2(n)$. Then for liner system.

$a_1x_1(n) + a_1x_2(n) \rightarrow a_1y_1(n) + a_2y_2(n)$ ……..(1)

For any nonlinear system the principle of superposition doesnot hold true and equation (i) is not satisfied.

Numerical:

For the following system, determine whether the system is liner or not.

(1) y(n) = 2x(n) +3

Solution:

$y_1(n) = 2x_1(n)+3$

$y_2(n) = 2x_2(n)+3$

Applying superposition principle suppose,

$x(n) = a_1x_1(n)+a_2x_2(n)$

Then,

$y_3(n) = 2[a_1x_1(n)+a_2x_2(n)]+3$

$= 2a_1x_1(n)+ 2a_2x_2(n)+3$

$y_4(n) = a_1y_1(n)+ a_2y_2(n)$

$= a_1 [2x_1(n)+3]+ a_2[2x_2(n)+3]$

$= = 2x_1(n)a_1 +2a_2x_2(n)+3a_1+3a_2$

Since,

$y_3(n) \neq y_4(n)$. The system is nonlinear.

(ii) $y(n) = x^2(n)$

Solution:

$y_1(n) = x_1^2(n)$

$y_2(n) = x_2^2(n)$

Applying input such that,

$x(n) = a_1x_1(n)+a_2x_2(n)$

The response will be,

$y_3(n) = [a_1x_1(n)+a_2x_2(n)]^2$

and again,

$y_4(n) = a_1y_1(n)+a_2y_2(n)$

$= a_1x_1^2(n) + a_2y_2^2(n)$

$\therefore y_3(n) \neq y_4(n)$ The system is nonlinear.

Shift invariance:-

A system is shift invariant, if the input output relationship doesnot vary with shift. In other words for a shift invariant system shift in the input signal results in corresponding shift in output. Mathematically,

$x(n) \longrightarrow y(n)$

Which means that y(n) is the response for x(n). If x(n) is shifted by n_0, then output y(n) will also be shifted by same shift n_0 i.e

$x(n-n_0) \longrightarrow y(n-n_0)$

Where n_0 is an integer.

If the system doesnot satisfy above expression, then the system is called shift variant system.

The system shifting both linearly and time invariant properties are popularly known as liner time invariant system or simply LTI systems.

Numerical:

Check whether the system are shift invariant or not.

(i) $y(n) = x^2(n)$

Solution:

Let us shift in input by n_0, then the output will be,

$$y_1(n) = x^2(n - n_0)$$
$$\therefore y_1(n) = y(n - n_0)$$

Hence the system is shift invariant system.

(ii) y(n) = x(mn)

Let us shift in input by n_0 then, response will be,

$y_1(n) = x(mn - n_0)$

and the shift in output by n_0 will yields.

$y(n - n_0) = x[mn - mn_0]$

$\therefore$ The system is shift variant.

Causality:-

A system is causal of the response does not begin before the input function is applied. This means that of input is applied at n= n_0 , then for causal system output will depend on values of input x(n) for $n \le n_0$. Mathematically,

$y(n_0) = T[x(n), n \le n_0]$ ……..(i)

The response of causal system to an input doesnot depend on future values of that input but depends only on present or past values of input.

On the other hands, of the response of the system to an input depends on future values of the input, the system is noncausal. A non causal system doesnot satisfy equation (i).

Causal system are physically realizable whereas non causal system cannot be implemented practically. There is no system possible practically which can produce its output before input is applied.

The equation,

$y(n) = x(n-1)$ describes the causal system and

$y(n) = x(n) - x(n+1)$ describe the non causal system.

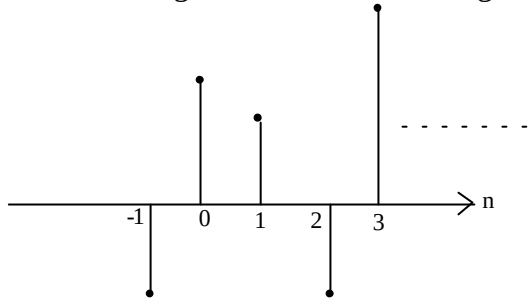
Memory less system:-

A system is referred to as memory less if the o/p y(n) at every value of n depends only on the i/p x(n) at the same value of n.

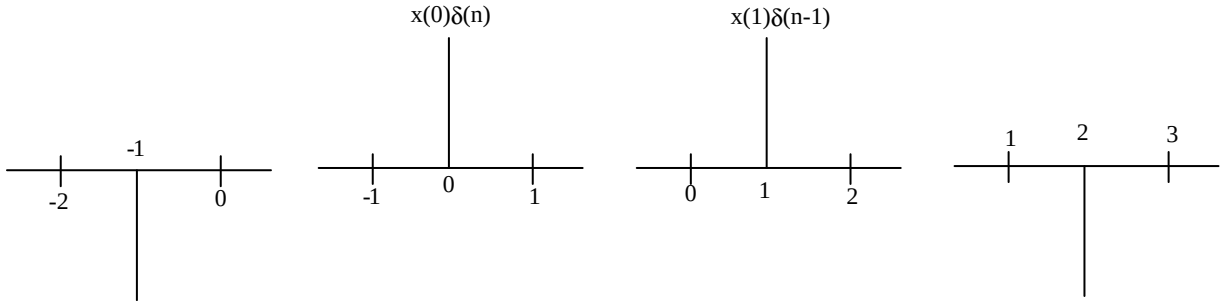
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*** Response of LTI system to Arbitrary input convolution solution:-**

Consider any arbitrary discrete time signal x[n] as shown in fig:



A discrete time sequence signal be represented by a sequence of individual impulses as shown in figure:



We can write,

X(n) = ……+x(-1)δ(n+1) +x(0) δ(n) +x(1) δ(n-1) +x(2) δ(n-2) + ……..

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \qquad(i) \text{ (convolution sum)}$$

Suppose h(n) is the o/p of LTI system when δ(n) is i/p. Therfore, the o/p for i/p x(-1) δ(n+1) is x(-1) h(n+1). Then the o/p y(n) for the i/p x(n) given in equation (i) will be,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad(ii)$$

Symbolically,

y(n) = x(n)* h(n) ……(iii)

y(n) = h(n)*x(n) …….(iv)

<u>Numericals:</u>

* The impulse response of invalid time response is: h(n) = {1, 2, 1, -1} x(n) = {1,2,3, }

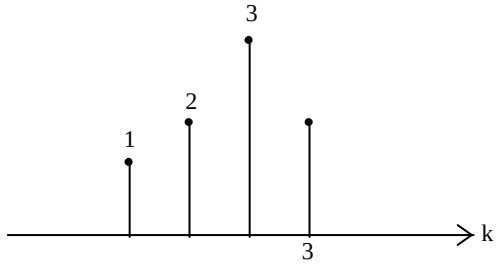
Solution:

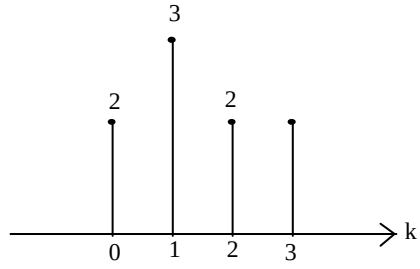
The response of the LTI system is ,

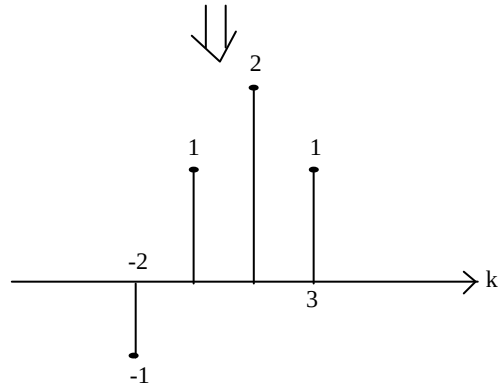
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

For n = 0 ,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$



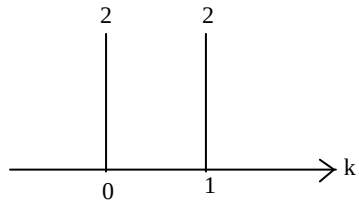




$$\sum_{k=0}^{1} x(k)h(-k)$$

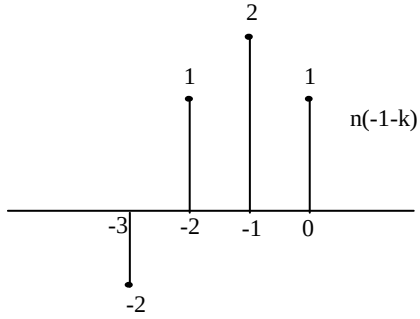
$$= x(0)h(0) + x(1)h(-1)$$

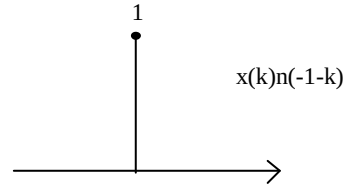
$$= 2 + 2 = 4$$



For n = -1

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$





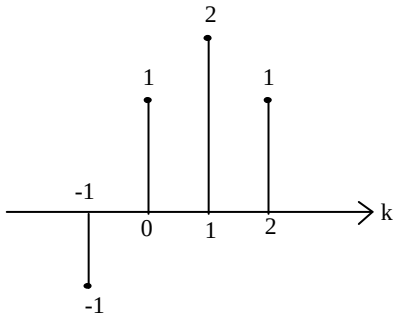
For n = -2

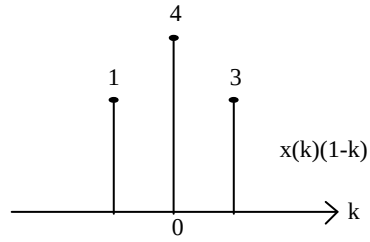
$$Y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$

Again testing for +ve side:

For n= 1

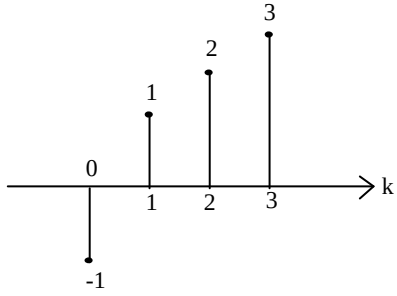
$$Y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 1+4+3 = 8$$

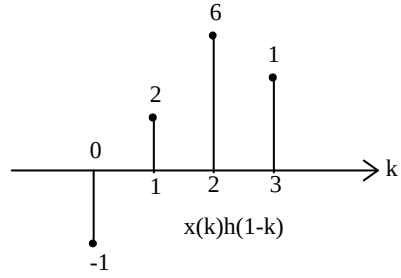




For n = 2

$$Y(n) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = -1+2+6+1 = 8$$





For n = 6 = 0

Y(n) = { …….0, 1, 4, 8, 8, 3,-2, -1, 0 …….)

Figure:

* *Determine the o/p y(n) of relaxed linear time invariant system with impulse response:*

$h(n) = a^n u(n)$, $|u| < 1$

When the i/p is unit step sequence ie;

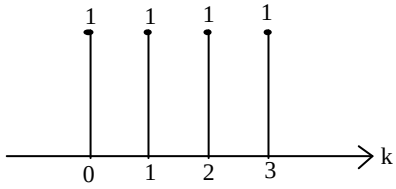
$x(n) = u(n)$

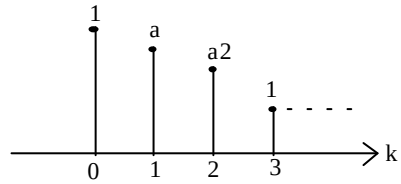
Solution:-

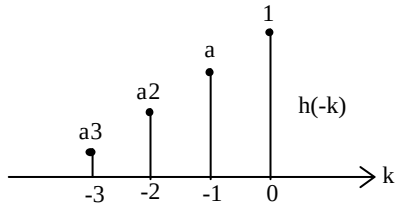
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

For n = 0 ,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

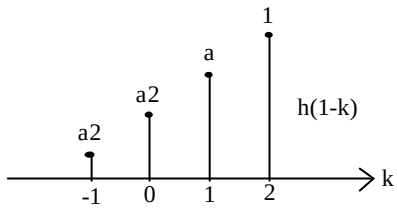


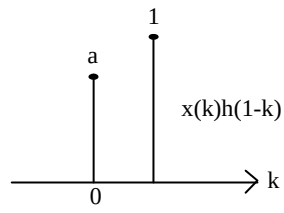




For n =1:

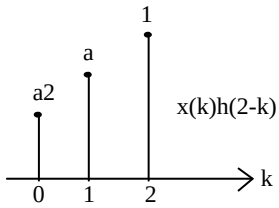
$y(1) = \sum_{n=-\infty}^{\infty} x(k)h(1-k) = 1+a$





For n = 2:

$y(2) = \sum_{k=-\infty}^{\infty} x(n)h(2-n) = 1+a+a^2$



$S = \frac{a(1-r^n)}{1-r}$

For n > 0

$y(n) = 1+a^2 + \ldots\ldots+a^n$

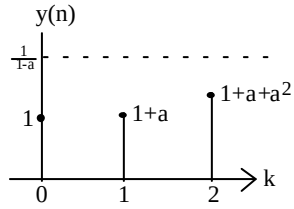
$= \frac{1-a^n}{1-a}$

For n < 0

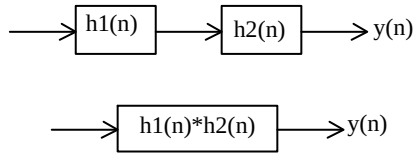
y(n) = 0

$y(\infty) = \lim n \to \infty \; y(n) = \lim n \to \infty \frac{1-a^{n+1}}{1-a}$

$= \frac{1}{1-a}$



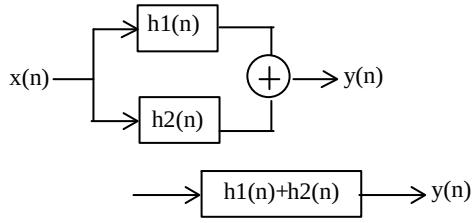
*** Interconnection of LTI system:-**



When two LTI systems with impulse response $h_1(n)$ and $h_2(n)$ are in cascade form the overall impulse response for the cascaded 2 impulse system will be,

$h(n) = h_1(n) * h_2(n)$ ……(i)

* The parallel combinations of LTI systems and equivalent system is shown below:



Determine the impulse response for the cascade of two LTI systems having impulse responses.

$h_1(n) = (1/2)^n u(n)$

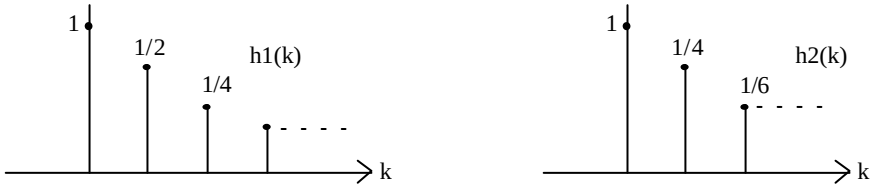
$h_2(n) = (1/u)^n u(n)$

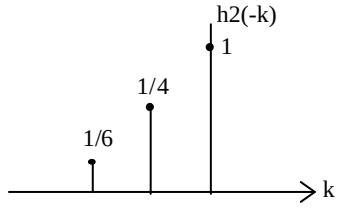
Solution:-

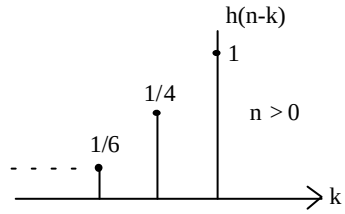
The overall impulse response is

$h(u) = h_1(n) * h_2(n)$

$$= \sum_{k=-\infty}^{\infty} h_1(k) h_2(n-k)$$







$$= \sum_{k=0}^{n} (1/2)^k (1/4)^{n-k}$$

$$= (1/4)^n \sum_{k=0}^{n} (1/2)^k (1/4)^{-k}$$

$$= (1/4)^n \sum (2)^k$$

$$= (1/4)^n \left(\frac{2^{n+1}-1}{2-1} \right)$$

$$= (1/4)^n (2^{n+1}-1)$$

$$= (1/2)^n [\, 2 - (1/2)^n] \;, n \geq 0$$

Note:- If we have 'L' LTI system is cascade with impulse responses $h_1(n)$ and $h_2(n)$ …….$h_L(n)$, the impulse response of equivalent LTI system is

$h(n) = h_1(n) * h_2(n) * h_3(n)$ ……..$* h_L(n)$

Discrete system response to discrete input:-

For a discrete-time system, consider as input sequence $x(n) = e^{jwn}$, for $-\infty < n < \infty$ the output of LTI system with impulse response h(n) is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{jw(n-k)}$$

$$= e^{jwn} \sum_{k=-\infty}^{\infty} h(k)e^{-jwk}$$

If we define,

$$H|e^{jw}| = \sum_{k=-\infty}^{\infty} h(k)e^{-jwk}$$

Then, $y(n) = H|e^{jw}| e^{jwn}$

$H|e^{jw}|$ describes the change in complex amplitude of complex exponential as a function of frequency w. $H(e^{jw})$ is called frequency response of the system.

In general $H(e^{jw})$ is complex and can be expressed in terms of its real and imaginary parts as:

$H(e^{jw}) = H_R(e^{jw}) + j\, H_I(e^{jw})$

Or , In terms of magnitude and phase as,

$H(e^{jw}) = |H(e^{jw})|e^{j\angle H(ejw)}$

Let,

$x(n) = A\cos(w_o n + \phi)$

$= A/2\ e^{jwon}.e^{j\phi} + A/2\ e^{-jwon}.\ e^{-j\phi}$

The response to $x_1(n) = A/2\ e^{j\phi}\ e^{jwon}$ is,

$y_1(n) = H(e^{jw})\ A/2\ e^{j\phi}\ e^{jwon}$

The response to $x_2(n) = A/2e^{-j\phi}\ e^{-jwon}$ is ,

$y_2(n) = H(e^{-jw})\ A/2\ e^{-j\phi}.\ e^{-jwon}$

Thus, total response is ,

$y(n) = A/2\ [H(e^{jwo})e^{-j\phi}.\ e^{jwon} + H(e^{-jwo})e^{-j\phi}.\ e^{-jwon}]$

$= A\ |H(e^{-jwo})|\ \cos(w_o n + \phi + \theta).$

Where, $\theta = \angle H(e^{jwo})$

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Stability: We defined arbitrary relaxed system as BIBO stable. If an only if o/p sequence y(n) is bounded for every bounded i/p x(n).

If x(n) is bounded their exist a constant M_x such that,

$$|x(n)| \le M_x < \infty$$

Similarly if o/p is bounded their exists a constant M_y such that

$|y(n)| \le M_y < \infty \quad for\ \ all\ n,$

Now given such a bounded input sequence x(n) to LTI system with convolution formula.

$$y(n) = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right|$$

$$\le \sum_{k=-\infty}^{\infty} |h(k)x(n-k)|$$

$$\le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

$$\le M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

From this expression we observe that the system is bounded if the impulse response of the system satisfied condition,

$$S_n = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

That is linear time invariant system is stable if its impulse response is absolutely summable.

Determine the range of value of the parameter a for LTI system with impulse response.

$h(n) = a^n\ u(n)$ is stable.

Solution,

$$S_n = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$$= \sum_{k=0}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |a_k|$$

$$= 1 + |a| + |a^2| + |a^3| + ...$$

$$= \frac{1}{a - |a|} \quad \text{Provided that } |a| < 1$$

Therefore the sytem is stable if $|a| < 1$ otherwise it is unstable.

Determien range of value of a,b for which LTI system with impulse response.

$h(n) = a^n \quad n \geq 0$

$= b^n \quad n < 0$ is stable.

$$S_n = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{-1} |b_k| + \sum_{k=0}^{\infty} |a_k|$$

The first sum coverage for $|a| < 1$. The second sum can be manipulated as,

$$\sum_{k=-\infty}^{-1} |b|^k = \sum_{k=1}^{\infty} \frac{1}{|b|^k}$$

$$= \frac{1}{|b|} + \frac{1}{|b|^2} + \frac{1}{|b|^3} + ...$$

$$= \frac{1}{|b|}\left(1 + \frac{1}{|b|^2} + \frac{1}{|b|^3} + ...\right)$$

$$= 1 + \beta + \beta^2 + \beta^3 + ...$$

$= \beta(1/1 - \beta)$ Provided that , $|\beta| < 1$ *or* $|b| > 1$

Hence the system is stable is both $|a| < 1$ *or* $|b| > 1$

Date: 2066/07/23

Determine whether the given system is BIBO stable or not.
y (n) = 1/3 [x(n) +x(n-1) +x (n-2)]

Solution:-

Assume that,

$|x(n)| < M_n < \infty$ for all n.

BIBO = Bounded input bounded output.

Then $|y(n)| = 1/3 [x(n) + x(n-1) + x(n-2)]$

$\leq 1/3 [|x(n)| + |x(n-1)| + |x(n-2)|]$

$\leq 1/3 [M_x + M_x + M_x]$

$\leq M_x$

Since, M_x is finite value , |y(n)| also finite. Hence the system is BIBO stable.

Determine whether the given system is BIBO stable or not.
Y(n) = r^nx(n) where r>1.

Solution:
Assume

$$|y(n)| = |r^n x(n)|$$
$$= r^n |x(n)|$$

With r >1, the multiplying factor r^n diverges for increasing n and he=nce o/p not bounded. Hence the system is BIBO unstable.

Nyquist Sampling theorem:-

A signal whose spectrum is band limited to B Hz (G(w) = 0 for |w| > 2πB) can be reconstructed exactly form its samples taken uniformly at the rate R > 2BHz (sample/sec) . In other words, minimum sampling frequency f_s = 2Bhz.

Consider a signal g(t) whose spectrum is band limited to Bhz. Sampling g(t) at the rate of f_s hz can be accomplished by multiplying g(t) by impulse δ_{Ts} (t) consisting of unit impulses repeating periodically every T_s second. Where, $T_s = 1/f_s$

Figure;

Trigonometric fourier series of impulse train,

$$\delta_{Ts}(t) = \frac{1}{T_s}[1 + \cos w_s t + 2\cos 2w_s t + 2\cos 3w_s t + ...]$$

Where, $W_s = 2\pi/T_s$

$$\bar{g}(t) = g(t)\,\delta_{Ts}(t)$$

$$= \frac{1}{Ts}[g(t) + 2g(t)\cos w_s t + 2g(t)\cos 2w_s t + 2g(t)\cos 3w_s t + ...]$$

Using modulation property,

$2g(t)\cos w_s t \leftrightarrow$ (F.T) $G(w-w_s)+G(w+w_s)$

$$\bar{G}(w) = \frac{1}{T_s}[G(w) + G(w - w_s) + G(w + w_s) + ...]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(w - nw_s)$$

If we want to reconstruct g(t) from g(t) bar we should be able to recovered G(w) form $\bar{G}(w)$. This is possible if there is no overlap between successive cycle of $\bar{G}(w)$. Figure (e) shows that this requires f_s greater then 2B.

From figure we see that g(t) can be recovered form sample $\bar{g}(t)$ by passing sampled signal through ideal low pass filter with bandwidth B hz.

Date:2066/7/26

Sampling of analog signals:

There are many ways to sample analog signal. We limit our discussion to periodic or uniform sampling which is he types of sampling used most often in topic in practice . This is described by the relation.

$X(n) = x_a(nT), \quad -\infty < n < \infty$

Where x(n) is discrete-time signal obtained by talking samples of analog signal $x_n(t)$ every T second.

T = sampling period or sample interval.

$F_s = 1/T$ = damping frequency or sampling rate (sample/sec or Hz)

$t = nT = n/F_s$

In general the smapling of continuous time sinusoidal signal

$x_a(t) = A\cos(2\pi f_0 t + \theta)$

With sampling rate $f_s = \frac{1}{T}$ results discrete time signal.

$x(n) = A\cos(2\pi f t + \theta)$

Where, $f = \frac{f_0}{f_s}$ = Relative frequency of sinusoid.

Q. Consider the analog signal $x_a(t) = 3\cos 100\pi t$

a) Determine the minimum sapling rate require to avoid aliasing.

b) Suppose that the signal is sampled at the rate F_s= 200 hz What is the discrete time signal obtained after sampling.

c) Suppose that the signal is sample at the rate F_s= 75 hz what is the discrete time signal obtained after sampling.

d) What is the frequency 0< f< F_s/2 of a sinusoids that yields samples identical to those obtained in part c.

Solution:-

$X_k(t) = 3\cos 100\pi t$

$2\pi f_0 = 100\pi$

$F_0 = 50$ hz

b) Mininum sampling rate = 100 hz

c) F_s= 200 hz

d) F_s = 75 hz

$X(n) = 3\cos(100\pi n/f_s) = 3\cos(100\pi n/75) = 3\cos(4\pi n3)$

$= 3\cos 2\pi(2/3)n$

$=3\cos 2\pi(1 -1/3)n$

$= 3\cos 2\pi(1/3)n$

e) F_s = 75 hz , f= 1/3

$f = F_0/F_s$, $F_0 = f F_s$ = 25 hz

$y(t) = 3\cos 50\pi t$

Q. Consider the analog signal at $x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. What is the Nequest rate for this signal.

Solution: -

Frequency present in the signal,

$F_1 = 25hz$, $F_2 = 150$ hz , $F_3 = 50$ hz

$F_{max} = 150$ hz

Nyquest rate = $2F_{max} = 300$ hz.

Q. Consider the analog signal at $x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 2000\pi t$.

a) What is the Nyquest rate for the signal.

b) Assume now that we sample the signal using sampling rate $F_s = 5000$ samples per second. What is the discrete time signal obtained after sapling.

Solution:

a) $F_1 = 1khz$ $F_2 = 3khz$ $F_3 = 6khz$

Nyquest rate = 12 khz

b) $F_s = 5000$ hz $= 5khz$

$x(n) = 3\cos(2000\pi\, n/F_s) + 5\sin(6000\pi\, n/F_s) + 10\cos(12000\pi\, n/F_s)$

$= 3\cos(2\pi n/5) + 5\sin(6\pi n/5) + 10\cos(12\pi n/5)$

$= 3\cos(2\pi n/5) + 5\sin 2\pi(3/5)n + 10\cos 2\pi(6/5)n$

$= 3\cos(2\pi n/5) + 5\sin 2\pi(1-2/5)n + 10\cos 2\pi(1+1/5)n$

$= 3\cos(2\pi n/5) - 5\sin 2\pi(2/5)n + 10\cos 2\pi(1/5)n$

$= 13\cos(2\pi n/5) - 5\sin 2\pi(2/5)n$

Date: 2066/07/26

Chapter:- 2

Discrete Fourier transform:-

Frequency domain sampling: Discrete Fourier transform. (DFT)

Let us consider aperiodic discrete-time signal x(n) with Fourier transform

$$X(w) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn} \quad \text{.....(i)}$$

Suppose that we sample X(w) periodically in frequency at a spacing of δ(w) radian between the successive samples. Since X(w) is periodic with period 2π only samples in the fundamental frequency range are necessary. We take N equidistant sample in the interval 0 <= w <=2π . With sample spacing $\delta w = 2\pi/N$

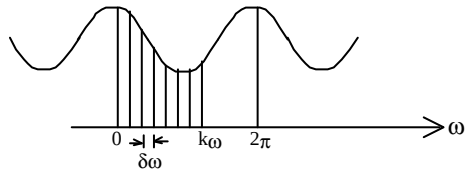


Fig: Freq. domain sampling.

If we evaluate, equation (i) we get,

W = 2π/N. K, k = 0 to N-1

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N}$$

Summing in equation (2) can be subdivided into infinite number of summations where each sum contains N terms Thus,

$$X\left(\frac{2\pi k}{N}\right) = ...+\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \sum_{n=2N}^{3N-1} x(n)e^{-j2\pi kn/N} + ...$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=\ln}^{lN+N-1} x(n)e^{-j2\pi kn/N}$$

If we change the index in the inner summation form n to n-ln and integrating the order of summation we obtained,

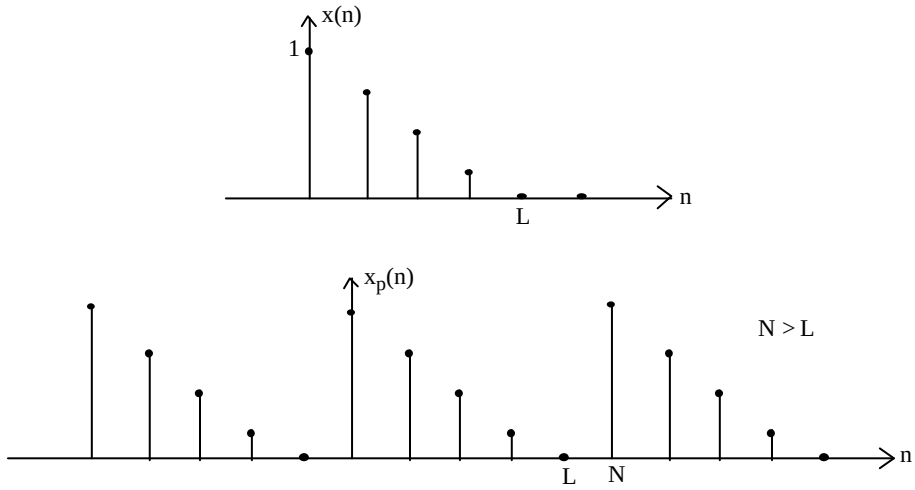
$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n-nl)\right]e^{-j2\pi kn/N} \quad \text{……..(3)}$$

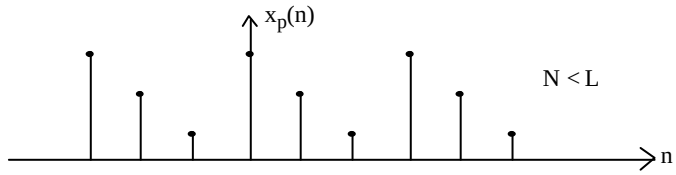
The signal

$$X_p(n) = \sum_{l=-\infty}^{\infty} x(n-nl) \quad \text{……..(4)}$$

Obtained by period representation of x(n) every N samples is clearly period with fundamental period N.

Since $x_p(n)$ is period extension of x(n) given by equation (4) it is clear that x(n) can be recovered from $x_p(n)$ if there is no alising in the time domain that is x(n) is limited to less than the period N of $x_p(n)$





In summary a final duration sequence x(n) of a length L as fourier transform $X(w) = \sum_{n=0}^{L-1} x(n)e^{-jwn}$

for 0 <= w <= 2π …….(5)

When we sample X(w) at equal space frequency $w_k = 2\pi K/N$, k = 0,1, 2 ……N-1

Where, N ≥ L,

The resultant sample are,

$$X(k) = X\left(\frac{2\pi K}{N}\right) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad \ldots\ldots\ldots\ldots (6)$$

Where, k = 0, 1, ……N-1

The relation in equation (6) is a formula for transforming sequence x(n) of length L <= N into a sequence of frequency samples {X(k)} of length L. Since the frequency samples are obtained by evaluating the fourier transform X(w) at a set of N equally spaced discrete frequencies. The relation in equation (6) is called discrete fourier transform (DFT) of x(n).

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$x_p(n)$ can be written as,

$$x_p(n) = \sum_{k=0}^{N-1} a_k e^{j2\pi kn/N} \quad n = 0, 1, \ldots\ldots\ldots N-1 \quad \ldots.(7)$$

With Fourier coefficient,

$$a_k = 1/N \sum_{k=0}^{N-1} x_p(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots\ldots..N-1 \quad \ldots\ldots\ldots(8)$$

Form equation (3) and (8)

$$a_k = 1/N \left(\frac{2\pi k}{}\right) \quad k = 0, 1, \ldots..N-1 \ldots\ldots.(9)$$

Therefore ,

$$x_p(n) = 1/N \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right)e^{j2\pi kn/N} \quad n = 0, 1, \ldots\ldots..N-1 \ldots\ldots.(10)$$

This relation allows us to recover the sequence x(n) from frequency sample.

$$x(n) = 1/N \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \quad n = 0, 1 \ldots\ldots.N-1 \ldots\ldots.(11)$$

This is called inverse DFT (IDFT) (

DFT as linear transformation:-

The formula for DFT and IDFT is given by

$$X(k) = \sum x(n) w_N^{kn} \quad , k = 0, 1 \ldots\ldots N\text{-}1 \quad \ldots\ldots..(1)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn} \quad n = 0, 1, \ldots\ldots N\text{-}1 \quad \ldots\ldots..(2)$$

Where $W_N = e^{-j2\pi/N}$ …….(3)
Let us define N-point vector x_N of signal sequence x(n) , n = 0, 1, ……N-1 and N-point vector X_n of frequency samples and N*n matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(n-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ X(N-1) \end{bmatrix} \quad W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^{N-1} \\ . & W_N^2 & W_N^4 & W_N^{2(N-1)} \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With there definition N-point DFT may be expressed in matrix
$X_N = W_N x_N$ ……..(4)
Where, W_N = matrix of liner transformation.
Equation (4) can be inverted by per multiplying both side by W_N^{-1}. Thus we can obtained $x_N = w_N^{-1} X_N$ …..(5)
Which is expression for IDFT
$x_N = 1/N\ W_N^{x}\ X_N$ ……..(6)
Where, W_N^* = complex conjugate of matrix W_N
$W_N^{-1} = W_N^*/N$ ……..(7)

Q. Compute DFT of 4-point sequence x(n) = { 0, 1, 2, 3}
Solution:-

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\pi kn/2}$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) = 6$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2} = x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2} = 6$$

X(2) = -2
X(3) = -2.4

By matrix method,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$x_N = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad X_4 = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$X_4 = W_4 * x_4$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Q. Compute IDFT for the frequency component X(k) = { 60, 0, -4, 0}

Solution:-

$x_N = (W_N^* / N) X_N \quad x_4 = (W_4^* / 4) X_4$

$$X_4 = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$x_4 = (W_4^* / 4) X_4$

$$= ¼ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 60-4 \\ 60+4 \\ 60-4 \\ 60+4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 56 \\ 64 \\ 56 \\ 64 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

Properties of DFT:-

The notation used to denote N-point DFT pair x(n) and X(k) as x(n) ← DFT→ X(k)

1) Periodicity:

If x(n) ← DFT→ X(k)

Where, x(n+N) = x(n) for all n.

X(k+N) = X(k) for all k.

Proof:

$$x(n+N) = 1/N \sum X(k) e^{j2\pi k(n+N)/N}$$

$$= 1/N \sum_{K=0}^{N-1} X(k) e^{j2\pi kn/N} e^{j2\pi k}$$

$$= x(n)$$

$$X(k+N) = \sum_{n=0}^{N-1} n(k) e^{j2\pi k(k+N)/N}$$

$$= \sum_{n=0}^{N-1} X(k) e^{j2\pi kn/N} e^{j2\pi m}$$

$$= X(k)$$

2) Linearity: -

If $x_1(n)$ ← DFT→ $X_1(k)$ and

$x_2(n)$ ← DFT→ $X_2(k)$ Then, for any real or complex valued constants a_1 and a_2

$x(n) = a_1x_1(n) + a_2x_2(n)$ ← DFT→ $X(k) = a_1X_1(k) + a_2X_2(k)$

Proof:-

$$a_1x_1(n) + a_2x_2(n) \leftarrow \text{DFT} \rightarrow \sum_{n=0}^{N-1} a_1x_1(n) + a_2x_2(n)\ e^{-j2\pi kn/N}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} + a_2 \sum_{n=0}^{N-1} x_2(n)\ e^{-j2\pi kn/N}$$

$$= a_1X_1(k) + a_2X_2(k)$$

Date:2066/08/01

Circular Symmetries of sequence:

N-point DFT of finite duration sequence x(n) of length $L \le N$ is equivalent to N-point DFT of periodic sequence $x_p(n)$ of period N which is obtained by periodically extending x(n) i.e

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

$$x_p(n) = x(n-k) = \sum_{l=-\infty}^{\infty} x(n - k - lN)$$

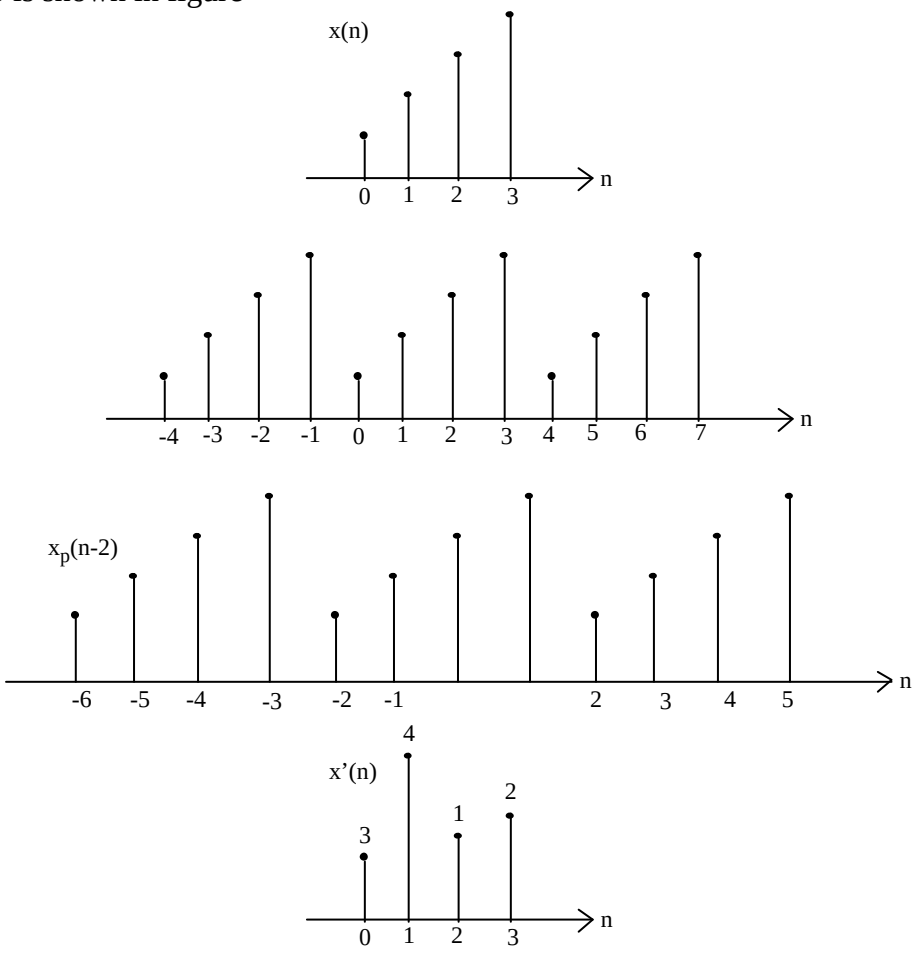
Finite duration sequence,

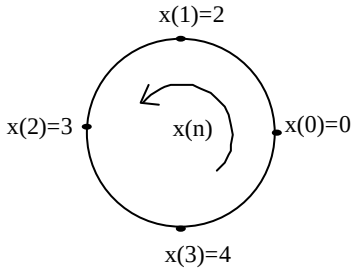
$x(n) = x_p(n) \quad 0 <= n <= N-1$

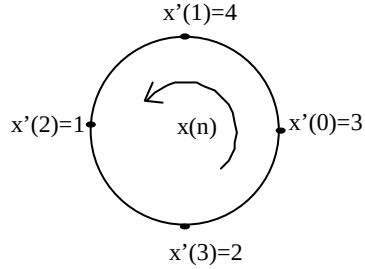
$= 0$ otherwise

Is related to original sequence by x(n) by circular shift. This relationship is shown graphically as follows.

This relationship is shown in figure







In general the circular shift of the sequence is represented as index modulo N. i.e $x(n) = x((n-k))_N$
For example,
K = 2, N = 4
That implies,
$x'(0) = x((-2))_4$
$x'(1) = x((-1))_4$
$x'(2) = x((0))_4$
$x'(3) = x((1))_4$
Hence x'(n) is shifted circularly by 2 units in time. where the counter clock wise direction is selected as the +ve direction. Thus we conclude that circular shift of N- point sequence is equivalent to linear shift of its period extension and vise versa.

Symmetry properties of DFT:-

Let us assume that N-point sequence x(n) and its DFT are both complex valued. Then the sequence can be respresented as

$x(n) = x_R(n)+jx_I(n)$ $0 <= n<=N-1$ ……..(1)

$X'(k) = X_R(k)+jX_I(k)$ $0<=k<=N-1$ …… …..(2)

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi n/N} \qquad k = 0, 1, \ldots\ldots\ldots N-1$$

$$= \sum_{n=0}^{N-1} (x_R(n) + jx_I(n))(\cos 2\pi kn/N - j\sin 2\pi kn/N)$$

$$= \sum_{n=0}^{N-1} (x_R(n)\cos 2\pi kn/N + x_I(n)\sin 2\pi kn/N) - j\sum_{n=0}^{N-1} (x_R(n)\sin 2\pi kn/N - x_I(n)\cos 2\pi kn/N)$$

$= X_R(k)+jX_I(k)$

$$X_R(k) = = \sum_{n=0}^{N-1} (x_R(n)\cos 2\pi kn/N + x_I(n)\sin 2\pi kn/N) \quad \ldots\ldots..(3)$$

$$X_I(k) = \sum_{n=0}^{N-1} (x_R(n)\sin 2\pi kn/N - x_I(n)\cos 2\pi kn/N) \quad \ldots\ldots\ldots(4)$$

Similary,

$x_R(n) = 1/N \sum_{k=0}^{N-1} (X_R(k)\cos 2\pi kn/N - X_I(k)\sin 2\pi kn/N)$(5)

$x_I(n) = 1/N \sum_{k=0}^{N-1} (X_R(k)\sin 2\pi kn/N + X_I(k)\cos 2\pi kn/N)$(6)

1) Real valued signal:-

If x(n) is real,
X(N-k) = X*(k) = X(-k)
Proof:-

$$X(N-k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi(N-k)n/N}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n/N} e^{-j2\pi n}$$

=X(-k) = X*(k)

2) Real and even sequence:-

x(n) = x(N-n)
Than, $X_I(0)$ = , DFT reduces to

$$X(k) = \sum_{n=0}^{N-1} x(n)\cos 2\pi kn/N \qquad 0<= k<=N-1$$

IDFT Reduces to ,

$$x(n) = 1/N \sum_{k=0}^{N-1} X(k)\cos 2\pi kn/N \qquad 0<=n<=N-1$$

3) Real and odd sequence:-

If x(n) = x(N-n) 0<=n<=N-1
Than, $X_R(k) = 0$, DFT reduces to ,

$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin 2\pi kn/N \quad , 0 <=k<= N-1$$

IDFT reduces to,

$$x(n) = j\ 1/N \sum_{k=0}^{N-1} X(k)\sin 2\pi kn/N \quad 0 <= n<= N-1$$

4) Purely Imaginary sequence:-

$x(n) = jx_I(n)$

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin 2\pi kn/N$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos 2\pi kn/N$$

If $x_I(n)$ is odd, then $X_I(k) = 0$ and hence X(k) is purely real.
If $x_I(n)$ is even then, $X_R(k) = 0$ and hence X(k) is purely imaginary.

Multiplication of two DFTs and circular convolution:-

Support we have two finite duriaotn sequences of length N, $x_1(n)$, $x_2(n)$. Their respective DFTs are,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \quad k = 0, 1, \ldots\ldots N-1 \quad \ldots\ldots\ldots (1)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \quad k = 0, 1, \ldots\ldots\ldots N-1 \quad \ldots\ldots\ldots (2)$$

If we multiply two DFTs together the result in DFT say $X_3(k)$ of a sequence $x_3(n)$ of length N. Let us determine the relationship between $x_3(n)$ and the sequence $x_1(n)$ and $x_2(n)$.

We have $X_3(k) = X_1(k)X_2(k)$

$$x_3(n) = 1/N \sum_{k=0}^{N-1} X_3(k) e^{j2\pi kn/m}$$

$$x_3(n) = 1/N \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j2\pi kn/m}$$

$$= 1/N \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/n} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N} \right] e^{-j2\pi kn/N}$$

$$= 1/N \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{-j2\pi k(m-n-l)/N} \right]$$

Now,

$$\sum_{k=0}^{N-1} a^k = N \qquad a = 1$$

$$= \frac{1-a^N}{a-a}, \quad a \neq 1$$

Where, $a = e^{j2\pi(m-n-l)/N}$

$$\sum_{k=0}^{N-1} a^k = N \qquad -l = m - n + pN = ((m-n))_N \quad p \text{ is integer}$$

$$= 0, \quad otherwise$$

Hence,

$$x_3(m) = 1/N \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N . N \qquad m = 0, 1, \ldots..N-1$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N$$

The expression has the form of convolution sum. The convolution sum involves index $((m-n))_N$ is called circular convolution. Thus we conclude that multiplication of DFT of two sequences is equivalent to the circular convolution of two sequences in time domain.

Q. perform the circular convolution of the following two sequences.
$x_1(n) = \{2, 1, 2, 1\}$
$x_2(n) = \{1, 2, 3, 4\}$

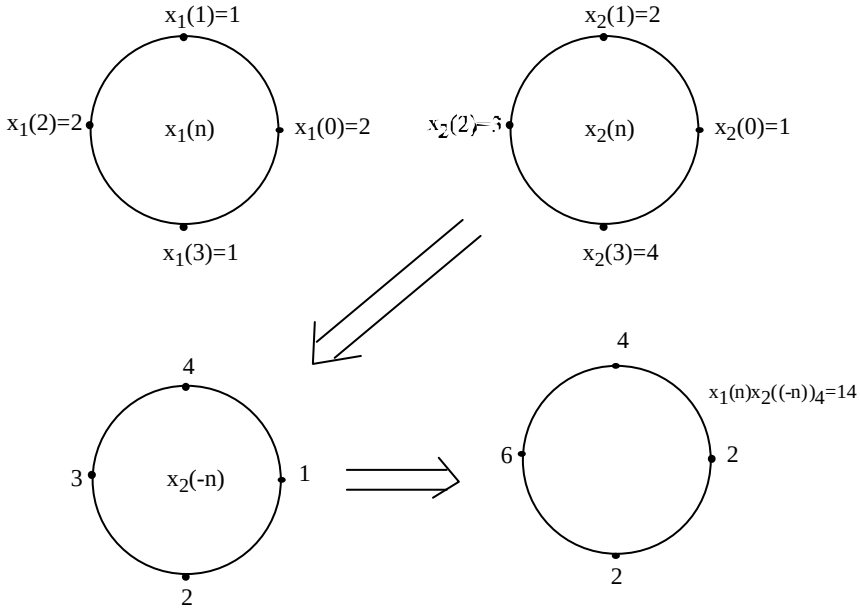
Solution,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N$$

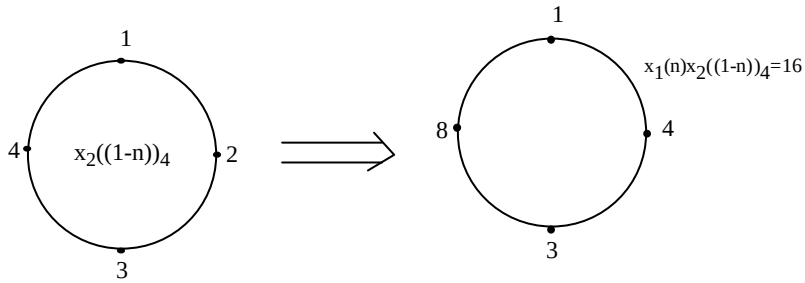
N = 4,

$$x_3(m) = \sum_{n=0}^{3} x_1(n)x_2((m-n))_4$$

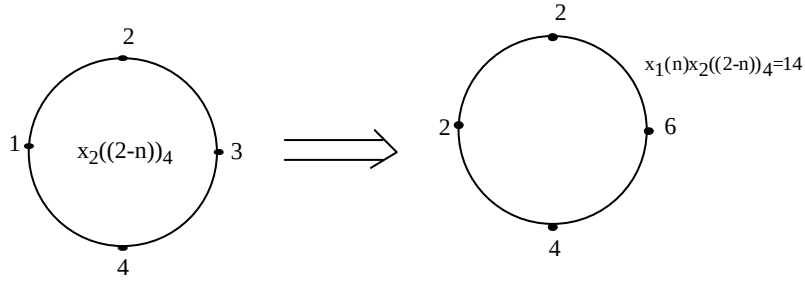
$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))_4 = 14$$



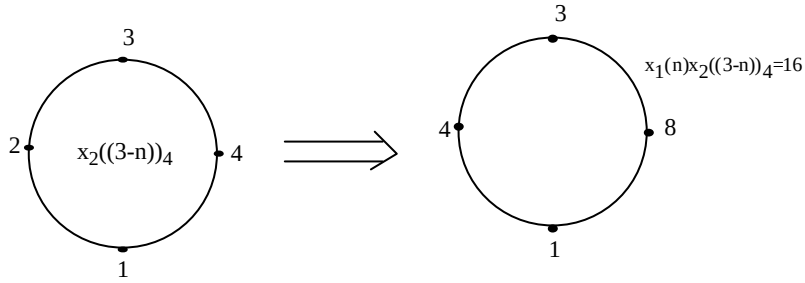
$$x_3(1) = \sum_{n=0}^{3} x_1(n)x_2((1-n))_4 = 16$$



$$x_3(2) = \sum_{n=0}^{3} x_1(n)x_2((2-n))_4 = 14$$



$x_3(3) = \sum_{n=0}^{3} x_1(n)x_2((3-n))_4 = 16$



$x_3(n) = \{ 14, 16, 14, 16\}$

Solution by DFT and IDFT method:

$$X_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

IDFT:

$x_3 = (W_4^*/4)* X_3$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

Parseval's theorem:-

For complex valued sequence x(n) and y(n)

x(n) ← DFT → X(k)

y(n) ← DFT → Y(k)

Then,

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

Proof:-

$$\sum_{n=0}^{N-1} x(n)y^*(n) = r^{xy}(0)$$ Circular cross correlation sequence.

$$r^{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} R_{xy}(k)e^{j2\pi kl/N}$$

$$r^{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)e^{j2\pi kl/N}$$

$$r^{xy}(0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

In Special case, y(n) = x(n)

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \sum_{n=0}^{N-1}|x(n)|^2 = \frac{1}{N}\sum_{n=0}^{N-1}|X(k)|^2$$

Which expresses the energy is finite duration sequence x(n) in term of frequency component {X(k)}

Date: 2066/08/04

Fast Fourier Transform (FFT):-

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{j2\pi k/N}$$

The complex multiplication in direct computation of DFT is N^2 and by FFT complex multiplication in $N/2 \log_2^N$. When number of points is equal to 4, the complex multiplication in direction computation of DFT is 16 and for FFT its value is 4. Hence the increment factor is 4.

Radix-2 FFT algorithm:-

If x(n) be the discrete time sequence then its DFT is given by $X(k) = \sum_{n=0}^{N-1} x(n) w_N^{kn}$ k = 0, 1,….N-1

Divide N-point data sequence into two N/2 data sequence $f_1(n)$ and $f_2(n)$ corresponding to the even number and odd number samples of x(n) respectively.

$f_1(n) = x(2n)$

$f_2(n) = x(2n+1)$ n = 0, 1, ……N/2-1

Thus $f_1(n)$ and $f_1(n)$ are obtained by decimating x(n) by a factor of 2 and hence the resulting FFT algorithm is called decimation in time algorithm.

$$X(k) = \sum_{n\ even}^{N-1} x(n) w_N^{kn} + \sum_{n\ odd}^{N-1} x(n) w_N^{kn}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) w_N^{k-2n} + \sum_{m=0}^{N/2-1} x(2m+1) w_N^{k(2m+1)}$$

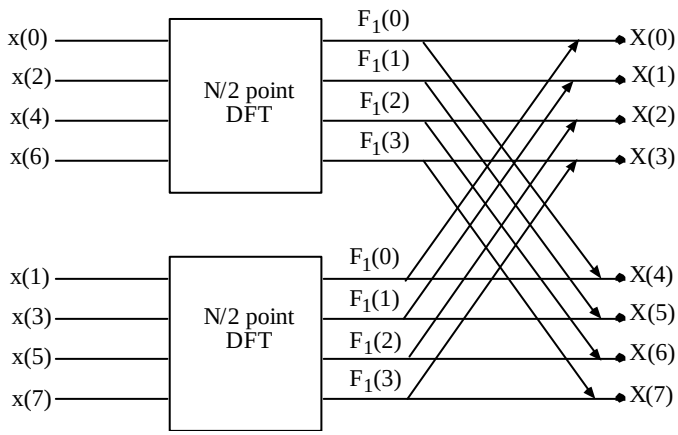
But

$$W_N^2 = \left(e^{-j2\pi n/2}\right)^2 = e^{-j2\pi/(n/2)} = W_{N/2}$$

$$X(k) = \sum_{m=0}^{N/2-1} f_1(m) w_{N/2}^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m) w_{N/2}^{km}$$

$X(k) = F_1(k) + W_n^k F_2(k)$

Where $F_1(k)$ and $F_2(k)$ are N/2 point DFT of sequences $f_1(m)$ and $f_2(m)$ respectively.



Having performed that DIT once, we can repeat the processor for each of sequence $f_1(n)$ and $f_2(n)$.

Thus $f_1(n)$ would result in two N/4 point sequence.

$v_{11}(n) = f_1(2n)$

$v_{12}(n) = f_1(2n+1)$, n = 0, 1,……N/4-1

And $f_2(n)$ would result.

$v_{21}(n) = f_2(2n)$

$v_{22}(n) = f_2(2n+1)$, n = 0, 1, …….N/4 -1

Then

$F_1(k) = v_{11}(k) + w_{N/2}^{k} v_{12}(k)$

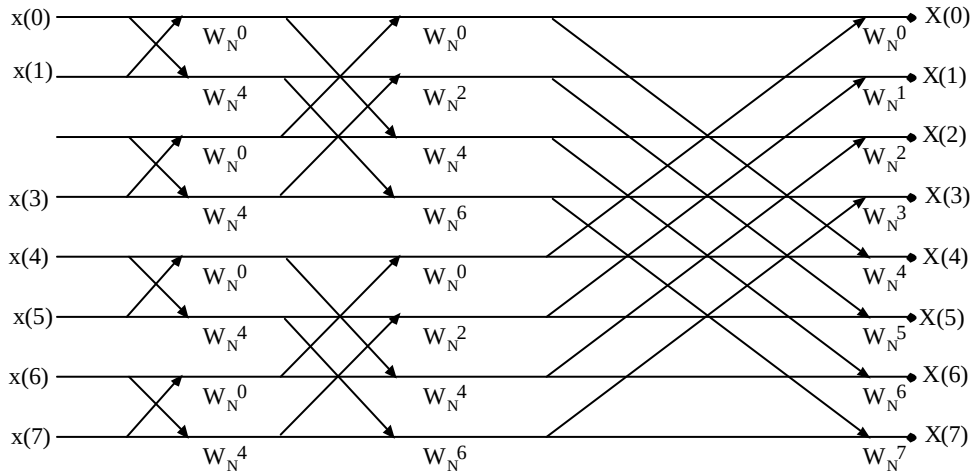


Fig: 8-point DIT FFT algorithm

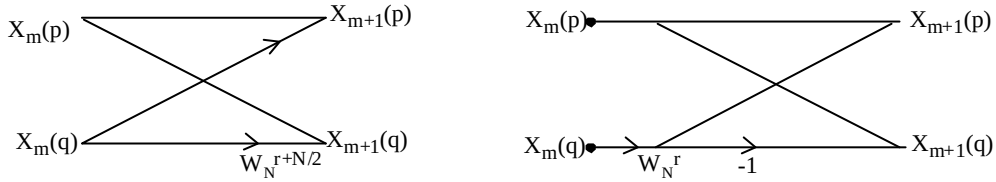


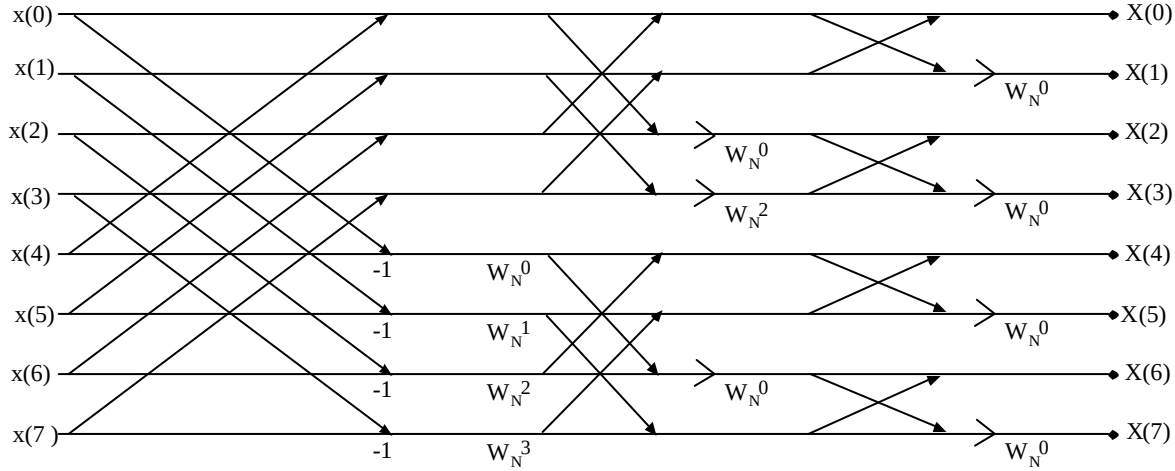
Fig: Basic butterfly computation in DIT FFT algorithm.

$X_{m+1}(p) = X_m(p) + W_N^r X_m(q)$

$X_{m+1}(q) = X_m(p) + W_N^{r+N/2} X_m(q)$

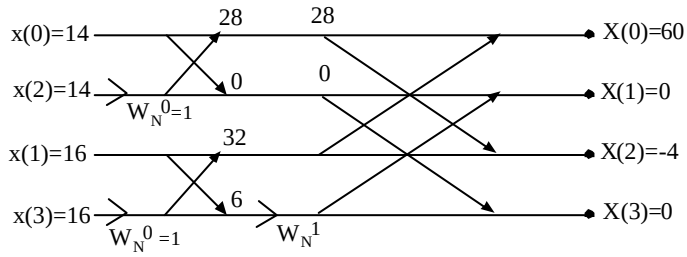
$= X_m(p) - W_N^r X_m(q) \qquad [W_N^{N/2} = -1]$

Compute 8 point DFT for the sequence x(n) = { 1,2, 3, 4,5 ,6 } . Using DITFFT algorithm or DIF FFT algorithm.

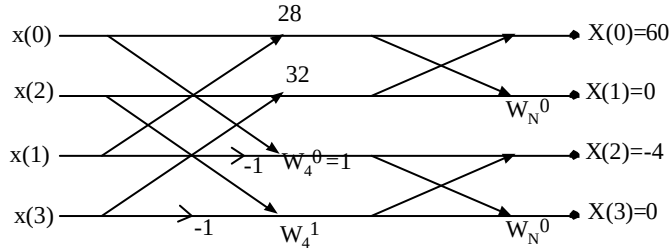


Date:2066/08/11

Q. Compute 4-point DFT for the sequence x(n) = { 14, 16,14, 16} using FFT algorithm.



Q. Compute 4 point DFT for the sequence x(n) = { 14, 16, 14, 16 } using FFT algorithm.



Date: 2066/08/11

Chapter: - 3

Z-transform:-

The z-transform of a discrete-time signal x(n) is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{.....(1)}$$

When z is complex variable.

The inverse procedure(i.e obtaining x(n) from X(z)) is called inverse z-transform . z-tranasform of a singal x(n) is denoted by

$X(z) = Z\{x(n)\}$ …….(2)

Where as the relationship between x(n) and X(z) is indicated by

$x(n) \leftarrow z \rightarrow X(z)$ ……..(3)

ROC (Region of convergence):

ROC of X(z) is the set of all values of z for which X(z) attins a finite values.

Determine z-transform of following finite duration signals.

$x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$= 1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}$

ROC: entire z-plane except z = 0 .

(2) $x_2(n) = \{2, 4, 5, 7, 0, 1\}$

$$X_2(z) = X(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} = 2z + 4z^2 + 5 + 7z^{-1} + z^{-3}$$

ROC: Entire z-plane. Except z = 0 and $z = \infty$

(3) $x_3(n) = \delta(n)$

$$X_3(n) = \sum_{n=-\infty}^{\infty} x_3(n) z^{-n}$$

$$= \left[\delta(n) z^{-n}\right]_{n=0}$$

$= 1$

ROC: Entire z-plane.

(4) $x_4(n) = \delta(n-u)$ $k > 0$

$$X_4(z) = X(z) = \sum_{n=-\infty}^{\infty} x_4(n) z^{-n} = 2z + 4z$$

$$\left[\delta(n-k) z^{-n}\right]_{n=k}^{z^{-k}}$$

ROC: Entire z-plane except z = 0

(5) $x_5(n) = \delta(n+k)$ $k>0$

$$X_5(n) = X(z) = \sum_{n=-\infty}^{\infty} x_5(n) z^{-n}$$

$$= \left[\delta(n-k) z^{-n}\right]_{n=k}$$

$= z^{-k}$

ROC: Entire z plane except z = 0.

ROC relationship to casusality:-

Let us express complex z-variable in polar form as.

$z = re^{j\theta}$ ……..(1)

Where r = |z| θ = angle z . Then X(z) can be expressed as,

$$X(z)|_{z= rej\theta} = X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \quad \text{……..(2)}$$

In the ROC of X(z) . $|X(z)| < \infty$

But,

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \quad \text{……..(3)}$$

Hence $|X(z)|$ is finite of the sequence $x(n)r^{-n}$ is absolutely summable.

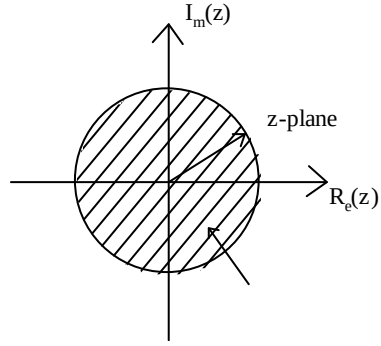
The problem of finding ROC for x(z) is equivalent to determining the range of values of r for which the sequence $x(n)\, r^{-n}$ is absolutely sum able.

$$|X(z)| \leq \sum_{n=-\infty}^{-1} |x(n) r^{-n}| + \sum_{n=0}^{\infty} |x(n) r^{-n}|$$

$$|X(z)| \leq \sum_{n=1}^{\infty} |x(-n) r^{n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \quad \text{…….(4)}$$

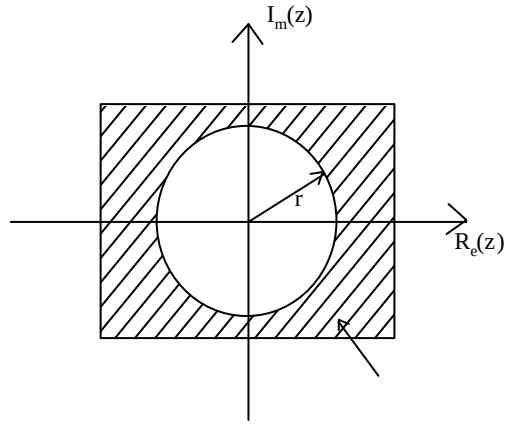
If x(z) converges in some region of complex plain both summation in equation (4) must be finite in that region.

If the first sum in equation (4) converges their must exists values of r small enough such that the product sequence $x(-n)\, r^{n}$, $1 <= n < \infty$ is absolutely summable. Therefore ROC for the first sum consists of all points in a circle of some radius r_1 where $r_1 < \infty$ as illustrated in the figure.

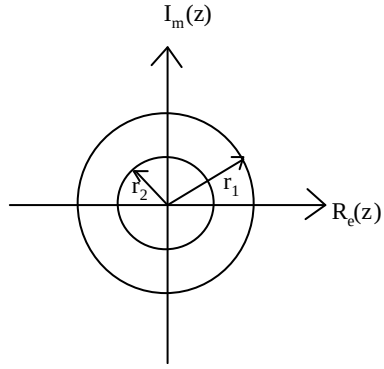


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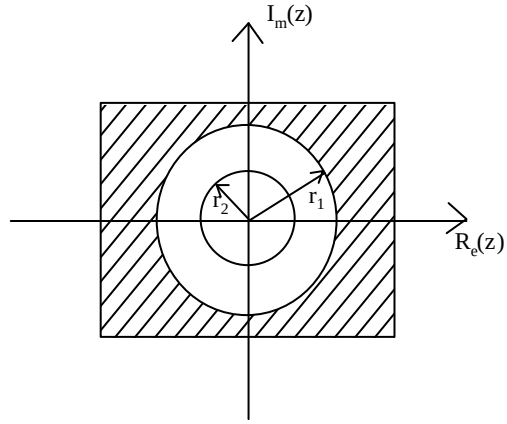
Now if the 2nd term in equation (4) converges there must exists values of r large enough such that product sequence $\frac{x(n)}{r^n}$, …….. hence ROC for second sum in equation (4) consists of all points outside a circle of radius $r > r_2$ as shown in fig.



Since the convergence of X(z) requires that both sums in equation (4) be finite it follows that ROC of X(z) is generally specified as the annual region in the z plane $r_2 < r < r_1$ which is common region where both sums are finite. Which is shown in figure.



If $r_2 > r_1$ there is no common region of convergence for the two sums and hence X(z) does not exist.



Numerical:

Determine z-transform of the signal.

$X(n) = \alpha^n u(n) = \alpha^n \quad , \; n => 0$

$= 0 \quad , \; n < 0$

Solution:

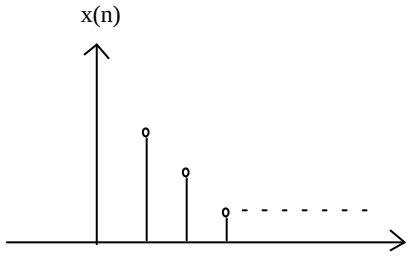
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

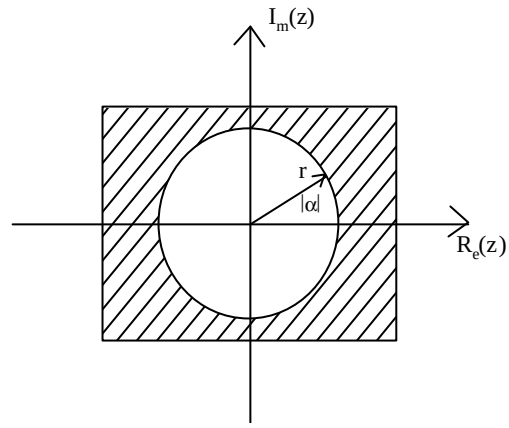
$$= \sum_{n=-\infty}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (\alpha z^{-1})^{-n} = 1 + (\alpha z^{-1}) + (\alpha z^{-1})^2 + ...$$

$$= \frac{1}{1 - \alpha z^{-1}} \quad for \ |\alpha z^{-1}| < 1$$

ROC: |z| > |α |





Determine z-transform of the signal.

X(n) = - α^n u(-n-1) = -α^n , n <= -1

= 0 , n > 0

Solution:-

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

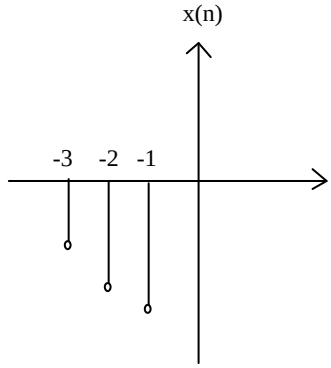
$$= \sum_{n=-\infty}^{-1} \left(-\alpha^n\right) z^{-n}$$

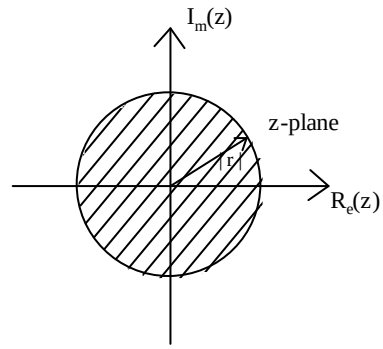
$$= -\sum_{l=1}^{\infty} \left(\alpha^{-l}\right) z^{l}$$

$$= -\sum_{l=1}^{\infty} \left(\alpha^{-l} z\right)^{l} = -\left[(\alpha^{-1} z) + (\alpha^{-1} z)^2 + ...\right]$$

$$= -\left(\frac{\alpha^{-1} z}{1 - \alpha^{-1} z}\right) \quad for \ \left|\alpha^{-1} z\right| < 1$$

$$= \frac{1}{1 - \alpha^{-1} z} \quad ROC: \ |z| < |\alpha|$$





Determine z-transform of the signal.

$X(n) = \alpha^n u(n) + b^n u(-n-1)$

Solution:

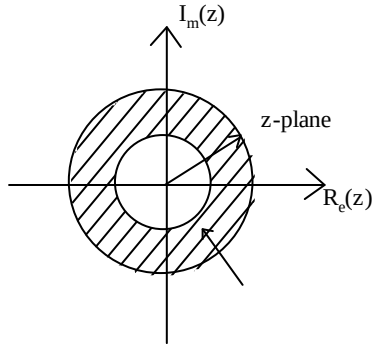
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

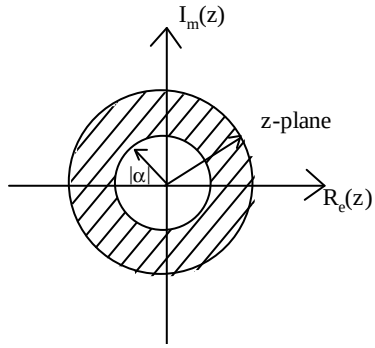
$$= \sum_{l=1}^{\infty} (b^{-1} z)^l + \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

The first power series converges if $|b^{-1}z| < 1$ i.e $|z| < |b|$ and second power series converges if $|\alpha z^{-1}| < 1$ i,e $|z| > |\alpha|$

Case I: $|b| < |\alpha|$



Case II: $|b| > |\alpha|$



$$X(z) = -\frac{1}{1-bz^{-1}} + \frac{1}{1-\alpha z^{-1}}$$
$$= \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}}$$

ROC: $|\alpha| < |z| < |b|$

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(1) linearity :

If , $x_1(n) \leftarrow z \rightarrow X_1(z)$

, $x_2(n) \leftarrow z \rightarrow X_2(z)$

Then $x(n) = a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_1X_2(z)$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= a_1X_1(z) + a_1X_2(z)$$

Determine z-transform and ROC of the signal

$x[n] = [\, 3(2^n) - 4(3^n)]$

Proof:

Let, $x_1(n) = 2^n u(n)$

$x_2(n) = 3^n u(n)$

Then,

$x(n) = 3x_1(n) - 4x_2(n)$

Accordign to linearity property.

$X(z) = 3X_1(z) - 4X_2(z)$

$X_1(z) = Z\{x_1(n)\} = \frac{1}{1-2z^{-1}}$ ROC: $|z| > 2$

$X_2(z) = Z\{x_2(n)\} = \frac{1}{1-3z^{-1}}$ ROC: $|z| > 3$

The intersection of ROC of $X_1(z)$ and $X_1(z)$ is $|z| > 3$.

Now

$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$ $|z| > 3$

Determine z-transform of the signals:

(a) $x(n) = (\cos w_o n)\, u(n)$

(b) $x(n) = (\sin w_o n)\, u(n)$

Solution:

$x(n) = (\cos w_o n)\, u(n)$

$= 1/2\left(e^{jw_0 n} + e^{-jw_0 n}\right)u(n)$

$= \left(\frac{1}{2}e^{jw_0 n}\, u(n) + \frac{1}{2}e^{-jw_0 n}\, u(n)\right)$

Using linearity property.

$X(z) = \frac{1}{2}Z\{e^{jw_0 n}\, u(n)\} + \frac{1}{2}Z\{e^{-jw_0 n}\, u(n)\}$

$Z\{e^{jw_0 n}u(n)\} = \frac{1}{1-e^{jw_0 n}z^{-1}} = \frac{1}{1-2z^{-1}}$ ROC: $|z| > 1$

$Z\{e^{-jw_0 n}u(n)\} = \frac{1}{1-e^{-jw_0 n}z^{-1}} = \frac{1}{1-2z^{-1}}$ ROC: $|z| > 1$

$X(z) = \frac{1}{2}.\frac{1}{1-e^{jw_0 n}z^{-1}} + \frac{1}{2}.\frac{1}{1-e^{-jw_0 n}z^{-1}}$

$= \frac{1-z^{-1}Cosw_0}{1-2z^{-1}\cos w_0 + z^{-2}}$ ROC: $|z| > 1$

(b) $x(n) = \frac{1}{2j}(e^{jw_0 n}e^{-jw_0 n})u(n)$

$X(z) = \frac{1-z^{-1}Sinw_0}{1-2z^{-1}\cos w_0 + z^{-2}}$ ROC: $|z| > 1$

(2) Time Shifting:-

If $x(n) \leftarrow z \rightarrow X(z)$

Then, $x(n-k) \leftarrow z \rightarrow z^{-k} X(z)$

Proof:

$x(n-k) \leftarrow z \rightarrow \sum_{n=-\infty}^{\infty} x(n-k)z^{-n}$

Put n-k = m

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-(m)}$$

$= z^{-k} X(z)$

By applying the time-shifting property determine z-transform of $x_2(n)$ and $x_3(n)$ form z-transform of $x_1(n)$ given as,

$x_1(n) = \{ 1,2, 5,7, 0, 1\}$, $x_2(n) = \{1,2, \overset{5}{\uparrow}, 7, 0, 1\}$
$x_3(n) = \{0,0,1,2,5,7,0,1\}$

Solution:

$X_2(n) = x_1(n+2)$, $x_3(n) = x_1(n-2)$

Now,

$X_2(z) = z^2 X_1(n)$

$X_1(z) = 1+2z^{-1}+5z^{-1}+7z^{-3}+z^{-5}$

$X_2(z) = z^2+2z+5+7z^{-1}+z^{-3}$

ROC: Entire z-plane except z = 0 and z = ∞

$X_3(z) = z^{-2} X_1(z)$

$= z^{-2}+2z^{-3}+5z^{-4}+7z^{-5}+z^{-7}$

Roc: entire z-plane except z = 0.

Determine the z-transform of the signal .

$$x(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$

Solution:

$$X(z) = \sum_{n=0}^{N-1} 1.z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-(N-1)} = N \quad \text{if } z = 1$$

$$= \frac{1-z^{-N}}{1-z^{-1}} \quad if \quad z \neq 1$$

Alternative method:-

$X(n) = u(n) - u(n-N)$

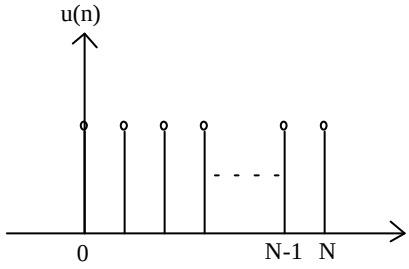
Using linearity and time shifting property.

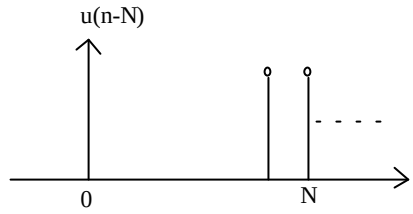
$X(z) = Z\{u(n)\} - Z\{u(n-N)\}$

$$= \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}}$$

$$= \frac{1-z^{-N}}{1-z^{-1}} \quad \text{ROC: } |z| > 1$$

Next method





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(3) Scaling in z-domain:-

If $x(n) \leftarrow z \rightarrow X(z)$ ROC, $r_1 < |z| < r_2$

Then $a^{-1}x(n) \leftarrow z \rightarrow X(a^{-1}z)$ ROC: $|a|r_1 < |z| < |a|r_2$

For any real constant a real or complex.

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(a^{-n}z)^{-n} = X(a^{-1}z)$$

ROC : $r_1 < |a^{-1}z| < r_2$

$$|a|r_1 < |z| < |a|r_2$$

Determine z-transform of the signals.

(a) $x(n) = a^n(\cos w_o n)u(n)$

(b) $x(n) = a^n(\sin w_o n)u(n)$

Solution:

$x(n) = a^n(\cos w_o n)u(n)$

$x_1(n) = \cos w_o n\, u(n)$

$$X_1(z) = Z[\cos w_0 n\, u(n)] = \frac{1 - z^{-1}\cos w_0}{1 - 2z^{-1}\cos w_0 + z^{-1}} \quad \text{ROC: } |z| > 1$$

Then,

$X(n) = a^n x_1(n)$

$$X(z) = Z[a^n x_1(n)] = X_1(a^{-1}z) = \frac{1 - az^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-1}} \quad \text{ROC: } |z| > |a|$$

(b) $x(n) = a^n(\sin w_o n)u(n)$

$$X(z) = = \frac{az^{-1}\sin w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-1}} \quad \text{ROC: } |z| > |a|$$

(4) Time reversal:

If $x(n) \leftarrow z \rightarrow X(z)$ ROC, $r_1 < |z| < r_2$

Then $x(-n) \leftarrow z \rightarrow X(z^{-1})$ ROC: $1/r_2 < |z| < 1/r_1$

Proof:

$$Z[\{x(-n)\}] = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

Put , l = -n

$$= \sum_{l=-\infty}^{\infty} x(l)z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

$= X(z^{-1})$

ROC of $X(z^{-1})$ is $r_1 < |z| < r_2$ or, $1/r_2 < |z| < 1/r_1$
Note that ROC of x(n) is the inverse of that for x(-n)

Determine the z-transform of the signal.
x(n) = u(-n)
Solution:

$u(n) \leftarrow z \rightarrow \frac{1}{1-z^{-1}}$ $|z| > 1$

Using time reversal property.

$u(-n) \leftarrow z \rightarrow \frac{1}{1-z}$ $|z| < 1$

(5) Differentiation in z- transform

If $x(n) \leftarrow z \rightarrow X(z)$

Then $n\,x(n) \leftarrow z \rightarrow -z\frac{dX(z)}{dz}$

Proof:-

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

By differentiation,

$$\frac{dX(z)}{dz} = \sum_{n=\infty}^{-\infty} x(n)(-n)z^{-n-1}$$

$$= -z^{-1}\sum_{n=\infty}^{-\infty}\{nx(n)z^{-n}$$

$$= -z^{-1}Z\{nx(n)\}$$

$Z\{nx(n)\} = -z\frac{dX(z)}{dz}$ Note that both transform have same ROC.

Determine z-trnasform of the signal.
$x(n) = na^n u(n)$
Proof:
$x_1(n) = a^n u(n)$

$x_1(n) = \frac{1}{a - az^{-1}}$ ROC; $|z| > |a|$

$x(n) = n\, x_1(n)$

$X(z) = Z\{n\, x_1(n)\} = \frac{dX_1(z)}{dz}$

$= -z \frac{-1}{(1-az^{-1})^2} az^{-2}$

$= \frac{az^{-1}}{(1-az^{-1})^2}$ ROC: $|z| > |a|$

Determine the signal x(n) whose z-transform is given by
$X(z) = \log(1+az^{-1})$, $|z| > |a|$

Solution:

$$\frac{dX(z)}{dz} = \frac{1}{(1+az^{-1})} . -az^{-2}$$

$$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{(1+az^{-1})} .$$

$$= az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right]$$

Taking inverse z-transform.

$n\, x(n) = a(-a)^{n-1}\, u(n-1)$

$x(n) = (-1)^{n-1} . a^n/n\ u(n-1)$

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If $x_1(n) \leftarrow z \rightarrow X_1(z)$

$x_2(n) \leftarrow z \rightarrow X_2(z)$

Then

$X(n) = x_1(n).x_2(n) \leftarrow z \rightarrow X(z) = X_1(z)\, X_2(z)$

The ROC of X(z) is at least the intersection of the for $X_1(z)$ and $X_2(z)$.

Proof:

The convolution of $x_1(n)$ and $x_2(n)$ is defined as

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

z-transform of x(n) as,

$$X(n) = \sum_{n=-\infty}^{\infty} x(n)z^{-1}$$

$$X(n) = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}\right]$$

$$X(n) = \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}$$

$$X(n) = \sum_{k=-\infty}^{\infty} x_1(k)z^{-n}x_2(z)$$

$$X(n) = X_2(z)\sum_{k=-\infty}^{\infty} x_1(k)z^{-n}$$

= $X_2(z)X_1(z)$

Compute convolution x(n) of the signals

X(n) = { 1,-2, 1}

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

Inverse z-transform.

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \quad \text{.....(i)}$$

Multplyign both sides of (i) by z^{n-1} and integrate both sides over a closed contour within the ROC of X(z) which enclosed the origin .

Now,

$$\oint X(z)z^{n-1}dz = \oint_C \sum_{k=-\infty}^{\infty} x(k)z^{n-1-k}dz \text{.....(2)}$$

Where C denotes closed counter in the ROC of X(z) .

We can interchanged the order of integration and summation on right hand side of (2) .

$$\oint X(z)z^{n-1}dz = \sum_{k=-\infty}^{\infty} x(k)\oint_C z^{n-1-k}dz \text{.....(3)}$$

From Cauchy integral theorem ,

Figure:

$$\frac{1}{2\pi j}\oint_C z^{n-1-k}dz = \begin{cases} 1 & k = n \\ 0 & k \ne n \end{cases}$$

Then,

$$\oint X(z)z^{n-1}dz = 2\pi j x(n)$$

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \text{ ……..(4)}$$

7) Multiplication of two sequences:-

If $x_1(n) \leftarrow z \rightarrow X_1(z)$
$x_2(n) \leftarrow z \rightarrow X_2(z)$
Then

$$x(n) \leftarrow z \rightarrow x_1(z)x_2(z) \leftarrow z \rightarrow X(z) = \frac{1}{2\pi j}\oint_C X_1(u)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

Where C is sthe closed counter that encloses the origin and lies within the ROC of common to both $X_1(v)$ and $X_2(1/v)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)z^{-n}$$

Where,

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(v)v^{n-1}dv$$

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{z^*}{v}\right)v^{-1}dv$$

$$X(z) = \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{\infty} x_2(n)\left(\frac{z}{v}\right)^{-n}\right]v^{-1}dv$$

8) Parseval's theorem:-

If $x_1(n)$ and $x_2(n)$ are complex valued sequence, then,

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2{}^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$$

$$X(z) = \sum c_n z^{-n}$$

$X(n) = c_n$ for all n.
Where X(z) is rational, the expansion can be performed by long division.

Determine inverse z-transform of

$$X(z) = \frac{1}{1-0.5z^{-1}+0.5z^{-2}}$$

When
(a) ROC: $|z| > 1$
(b) ROC : $|z| < 0.5$

Solution:

(a) ROC: |z| >1 , x(n) is causal signal.

$1-0.5z^{-1}+0.5z^{-2}$) 1 ($1-0.5z^{-1}-0.25z^{-2}$

$1-0.5z^{-1}+0.5z^{-2}$

$0.5z^{-1}-0.5z^{-2}$

$0.5z^{-1}-0.25z^{-2}+0.25z^{-3}$

$-0.25z^{-2}-0.5z^{-3}$

$-0.25z^{-2}+0.5z^{-3}-0.125z^{-4}$

$0.375z^{-3}+0.125z^{-4}$

$-0.375z^{-3}+0.18175z^{-4}+0.18175z^{-5}$

$-0.0625z^{-4}+1.8175z^{-5}$

$X(z) = \{1+0.5z^{-1}-0.25z^{-2}-0.375z^{-3} \ldots\ldots..\}$

$x(n) = \{ 1, 0.5, -0.25, -3.75, \ldots\ldots..\}$

(b) ROC : |z| <0.5 x(n) is anticausal signal.

$0.5z^{-2}-0.5z^{-1}+1)$ 1 ($2z^{2}+2z^{3}-2z^{4}$

$1-z+2z^{2}$

$z-2z^{2}$

$z-z^{2}+2z^{3}$

$z^{2}-2z^{3}$

$-z^{2}+z^{3}-2z^{4}$

$-3z^{3}+2z^{4}$

$-3z^{3}+3z^{4}-6z^{5}$

$-z^{4}+6z^{5}$

$$X(z) = 2z^2 + 2z^3 - 2z^4 - 6z^5 \ldots\ldots..$$

$$= \ldots\ldots\ldots\ldots 6z^5 - 2z^4 + 2z^3 + 2z^2 + 0 + 0$$

$$\therefore x(n) = \{.....-6,-2,\ 2\ ,\ 0,\ 0\}$$

Initial value theorem:

$X(0) = \lim z \to \infty\ X(z)$

Final value theorem:-

$\text{Lim } n \to \infty\ x(n) = \lim z \to 1\ (z-1)\ X(z)$

The impulse response of relaxed LTI system is $h(n) = \alpha^n u(n)$ $|\alpha| < 1$. Determine the step response of the system where n tends to infinity.

Solution:-

$y(n) = x(n) * h(n)$

$x(n) = u(n)$

$h(n) = \alpha^n u(n)$

$Y(z) = X(z)\ H(z)$

$$X(z) = \left(\frac{1}{1-z^{-1}}\right)\left(\frac{1}{1-\alpha z^{-1}}\right)$$

Now,

Lim $n \to \infty$ y(n) = lim z $\to$1 (z-1)Y(z)

= lim z tends to 1 (z-1) $\frac{z^2}{(z-1)(z-\alpha)}$

= $\frac{1}{1-\alpha}$

System function (transfer function) of LTI System:-

The o/p of LTI system to an input sequence x(n) can be obtained by comparing the convolution of x(n) with unit sample of the system. i.e y(n) = x(n)*h(n)

We can express this relationship in z-domain as

Y(z) = X(z) H(z)

When, Y(z) = z-transform of y(n)

X(z) = z-transform of x(n)

H(z) = z-transform of h(n)

Now, H(z) = $\frac{Y(z)}{X(z)}$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

H(z) represent the z-domain characteristics of the system where as h(n) is corresponding time domain characteristics of the system. The transform H(z) is called the system function or transfer function.

General form of liner constant coefficient difference equation:-

The system is described by liner constant coefficient difference form.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \text{.....(1)}$$

Taking z transform on both sides,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \text{.....(2)}$$

Therefore a LTI system described by constant coefficient difference equation has a rational system function from this general form we obtained two important special forms:

1. if $a_k = 0$, for 1 <= k<= N . Equation (2) reduces to $H(z) = \sum_{k=0}^{M} b_k z^{-k}$ (all zero system) such a system has finite duration impulse response and it is called FIR system or moving average (MA) system.

2. On the other hand if $b_k = 0$, for 1 <= k<= n the system function reduces to

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$ …….(4) (all pole system) Due the presence of poles the impulse response of such a system is infinite in duration and hence it is IIR system.

Determine the system function and unit sample response of the system describe by the difference equation y(n) = ½ y(n-1) +2x(n).

Solution:

Taking z-transform on both sides,

$$Y(z) = 1/2\,Y(z) + 2X(z)$$

$$(1 - 1/2\, z^{-1})Y(z) = 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - 1/2\, z^{-1}}$$

By inversion,

$h(n) = 2\,(1/2)^n\, u(n)$

Date: 2066/09/16

Response of pole zero system with Non-zero initial condition:-

The difference equation,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$ ……(i)

Suppose that the signal x(n) is applied to the pole zero system at n = 0 . Thus the signal x(n) is assume to be a causal. The effects of all previous input signal to the system are reflected in the initial condition y(-1), y(-2) ……..y(-N). Since the input x(n) is causal and since we are interested in terminating the o/p signal y(n) for $n \geq 0$ we can use one sided z-transform which allows us to deal with initial condition.

Now,

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-1}\left[Y^+(z) + \sum_{n=1}^{k} y(-n) z^{-n} \right] + \sum_{k=0}^{m} b_k z^{-k} X^+(z)$$ ……..(2)

Since x(n) is causal, We can set $X^+(z) = X(z)$

$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} . X(z) - \frac{\sum_{k=0}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{-n}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= H(z)X(z) + \frac{N_o(z)}{A(z)}$$ ……..(3)

Where,

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(n) z^{n}$$ ……(4)

From (3) the o/p of the system can be sub divided into two parts. The 1st part is zero state response of the system defined in z-domain as $Y_{zs}(z) = H(z)X(z)$ …….. (5).

The second component corresponding to o/p resulting form initial condition. This o/p is zero input response of the system which is defined in z-domain as $Y_{zi}(z) = \frac{N_0(z)}{A(z)}$ ……(6)

Since the total response is the sum of the two o/p component which can be expressed in time domain by determining inverse z-transform of $Y_{zs}(z) = Y_{zi}(z)$. Seperately and adding the result.

$y(n) = y_{zs}(n) + y_{zi}(n)$ ……..(7)

Determine the unit step response of the system described by the difference equation $y(n) = 0.9y(n-1) – 0.81\ y(n-1) + x(n)$. Under the following initial condition.

(a) $y(-1) = y(-2) = 0$.

(b) $y(-1) = y(-2) = 1$.

Taking one-side z-trnasfrom.

$$Y^+(z) = 0.9z^{-1}\left[Y^+(z) + y(-1)z\right] - 0.8z^{-2}\left[Y^+(z) + y(-1)z + y(-2)z^{-2}\right] + X^+(z)$$

For $y(-1) = y(-2) = 0$

$$Y^+(z) = 0.9z^{-1}Y^+(z) - 0.81z^{-2}Y^+(z) + X^+(z)$$

$$\frac{Y^+(z)}{x^+(z)} = \frac{1}{1 - 0.9z^{-1} - 0.81z^{-2}}$$

$$\frac{Y_{zs}^+(z)}{z} = \frac{z^2}{(z^2 - 0.9z + 0.8z)(z-1)} = \frac{z^2}{(z - 0.9e^{j\pi/3})(z-1)}$$

$$\frac{z^2}{(z^2 - 0.9e^{j\pi/3})(z - 0.9e^{-j\pi/3})(z-1)} = \frac{k_1}{(z - 0.9e^{j\pi/3})} + \frac{k_1^*}{(z - 0.9e^{-j\pi/3})} + \frac{k_2}{z-1}$$

$k_1 = 1.098e^{-j5.53^o}$

$k_* = 1.098e^{j5.53^o}$, $k_2 = 1.1$

$$Y_{zs}^+(z) = \frac{1.098e^{-j5.53}}{(1 - 0.9e^{j\pi/3}z^{-1})} + \frac{1.098e^{j5.53}}{(1 - 0.9e^{-j\pi/3}z^{-1})} + \frac{1.1}{1 - z^{-1}}$$

By inversion,

$$y_{zs}(n) = \left[1.098e^{-j5.53}(0.9e^{j\pi/3})^n + 1.098e^{j5.53}(0.9e^{-j\pi/3})^n + 1.1\right] + y(n)$$

$$= \left[1.1 + 1.098(0.9)^n \cos(n\pi/3 - 5.53)\right]u(n)$$

(b) $y(-1) = y(-2) = 1$.

$$Y_{zi}(z) = \frac{N_o(z)}{A(z)}$$

$$N_o(z) = \sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(n)z^{-n}$$

$$N_o(z) = -\sum_{1}^{2} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{-n}$$

$$= \left[a_1 z^{-1} y(-1)z + a_2 z^{-2}(y(-1)z + y(-2)z^2)\right]$$

$$N_0(z) = -\left[-0.9\times 1 + 0.81 z^{-1} + 0.81\times 1\right] = 0.09 - 0.81 z^{-1}$$

$$Y_{zi}(z) = \frac{0.09 - 0.81 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

$$Y(z) = Y_{zs}(z) + Y_{zi}(z)$$

$$= \frac{1.099}{1 - z^{-1}} + \frac{0.568 + j0.445}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.568 - j0.445}{1 - 0.9 e^{j\pi/3} z^{-1}}$$

$$y(n) = \left[1.099 + 1.44(0.9)^n \cos(n\pi/3 + 38)\right] u(n)$$

Date: 2066/09/20

Determine well known fibanacci sequence of integer numbers is obtained by computing each term as the sum of two previous ones, the first few terms of the sequences are

1, 1, 2, 3, 5, 8 ……..
Determine a close form expression for the nth term of Fibonacci series.
Solution:-
Let y(n) be the nth term. Then,
Y(n) = y(n-1) + y(n-2) ……(i)
With initial condition ,
y(0) = y(-1)+y(-2) = 1
y(1) = y(0)+y(-1) = 1
y(-1) = 1-1 = 0, y(-2) = 1 .
Taking one sided z-tranzsform on both sides of (i).

$$Y^+(z) = z^{-1} Y^+(z) + y(-1) + z^{-2} Y^+(z) + Y(z) + y(-1) z^{-1}$$

$$Y^+(z) = z^{-1} Y^+(z) + z^{-2} Y^+(z) + 1$$

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$

$$\frac{Y^+(z)}{z} = \frac{z}{z^2 - z - 1} = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right) \cdot \frac{1}{z - \left(\frac{1+\sqrt{5}}{2}\right)} + \left(\frac{1-\sqrt{5}}{-2\sqrt{5}}\right) \cdot \frac{1}{z - \left(\frac{1-\sqrt{5}}{2}\right)}$$

$$Y^+(z) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right) + \frac{1}{1 - \left(\frac{1+\sqrt{5}}{2}\right) z^{-1}} - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right) \cdot \frac{1}{1 - \left(\frac{1-\sqrt{5}}{2}\right) z^{-1}}$$

Taking inverse z-transformation.

$$y(n)=\left[\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n\right]u(n)$$

Determine step response of the system y(n) = α y(n-1) + x(n) $-1 < \alpha < 1$
When initial condition is y(-1) = 1.

Solution:

Taking one sided z-transform,

$$Y^+(z)=\alpha\left[z^{-1}Y^+(z)+y(-1)\right]+X^+(z)$$

$$X^+(z)=Z\left[u(n)\right]=\frac{1}{1-z^{-1}}$$

Or , $Y^+(z)=\alpha z^{-1}Y^+(z)+\alpha+\frac{1}{1-z^{-1}}$

Or, $Y^+(z)(1-\alpha z^{-1})=\alpha+\frac{1}{1-z^{-1}}$

$$Y^+(z)=\frac{\alpha}{1-\alpha z^{-1}}+\frac{1}{(1-z^{-1})(1-\alpha z^{-1})}$$

$$Y^+(z)=\frac{\alpha}{1-\alpha z^{-1}}+\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-z^{-1}}\right)-\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-\alpha z^{-1}}\right)$$

Taking inversion,

$$y(n)=\alpha\alpha^n u(n).1\left(\frac{1}{1-\alpha}\right)u(n)-\left(\frac{1}{1-\alpha}\right)\alpha^n u(n)$$

$$=\left[\frac{\alpha^{n+1}(1-\alpha)+1-\alpha^{n+1}}{1-\alpha}\right]$$

$$y(n)=\left[\frac{1-\alpha^{n+2}}{1-\alpha}\right]u(n)$$

Determine the response of the system y(n) = 5/6. y(n-1) – 1/6 y(n-2) + x(n) to the input signal

$$x(n)=\delta(n)-\frac{1}{3}\delta(n-1)$$

Solution:

Taking z-transform in both sides.

$$Y(z)=\frac{5}{6}z^{-1}Y(z)-\frac{1}{6}z^{-2}Y(z)+X(z)$$

$$X(z)=1-\frac{1}{3}z^{-1}$$

$$Y(z)=\frac{5}{6}z^{-1}-\frac{1}{6}z^{-2}Y(z)+\left(1-\frac{1}{3}z^{-1}\right)$$

$$=\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)Y(z)=\left(1-\frac{1}{3}z^{-1}\right)$$

$$Y(z) = \frac{\left(1 - \frac{1}{3}z^{-1}\right)}{\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right)}$$

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Taking inverse z-transform,

$$y(n) = \left(\frac{1}{2}\right)^n u(n)$$

Causality and stability:-

A causal LTF system is one where unit sample response u(n) satisfies the condition h(n) = 0, for n < 0. We have also shown that ROC of z-transform of a causal sequence is exterior of a circle.

A necessary and sufficient condition for a LTI system to be BIBO stable is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. In turn this condition implies that H(z) must contain the unit circle within it's ROC.

Indeed, Since

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$|H(z)| \le \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right|$$

When evaluating on the unit circle i.e $|z| = 1$

$$|H(z)| \le \left| \sum_{n=-\infty}^{\infty} h(n) \right|$$

Hence if the system is BIBO, the unit circle is contained in the ROC of H(z).

A LTI system is characterized by the system function.

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

Specify the ROC of H(z) and determine h(n) for the following condition.

(a) The system is stable.
(b) The system is causal.
(c) The system is anticausal.

Solution:-

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

The system has poles at z = ½ and z = 3.

(a) Since the system is stable, its ROC must include the unit circle and hence it is ½ < |z| <3. Consequently h(n) is non-causal.

$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2.(3)^n u(-n-1)$ (The system is unstable).

(b) Since the system is causal , its ROC is |z| >3. In this case $h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$ (The system is unstable).

(c) Since the system is anticausal it's ROC is |z| <1/2 . In this case,

$h(n) = \left(\frac{1}{2}\right)^n u(-n-1) + 2.(3)^n u(-n-1)$. (The system is unstable).

Date: 2066/09/22

Transient and steady state response:

Determine the transient and steady state response of the system characterized by the difference equation y(n) = 0.5y(n-1) +x(n). When the input signal is x(n) = 10cos $\left(\frac{n\pi}{4}\right)u(n)$. The system is initially at a rest (i.e it is relaxed).

Solution:

Taking z-transform,

$$Y(z) = 0.5z^{-1}Y(z) + X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$$

$$Z(\cos w_o n\, u(n)) = \frac{1 - z^{-1}\cos w_o}{1 - 2z^{-1}\cos w_o + z^{-2}}$$

$$X(z) = 10\left[\frac{1 - z^{-1}\cos\frac{\pi}{4}}{1 - 2z^{-1}\cos\frac{\pi}{4} + z^{-2}}\right] = \frac{10\left(1 - \frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1 - \sqrt{2}z^{-1} + z^{-2}\right)}$$

$$Y(z) = \frac{1}{(1-0.5z^{-1})}.10\frac{\left(1 - \frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1 - \sqrt{2}z^{-1} + z^{-2}\right)}$$

$$= \frac{10(1 - \frac{1}{\sqrt{2}}z^{-1})}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

$$= \frac{63}{1-0.5z^{-1}} + \frac{6.78e^{-j2.870}}{1-e^{j\pi/4}z^{-1}} + \frac{6.78e^{j28.7}}{1-e^{-j\pi/4}z^{-1}}$$

The natural or transient response is

$y_{nr}(n) = 6.3(0.5)^n u(n)$ and forced or steady response is

$$y_{fr}(n) = \left[6.78e^{j28.7}e^{jn\pi/4} + 6.78e^{j28.7}e^{-jn\pi/4}\right]u(n)$$

$$= 13.56\cos\left(n\frac{\pi}{4} - 28.7\right)u(n)$$

Pole-zero diagram:-

$$H(z) = \frac{1 - z^{-1} - 2z^{-2}}{1 - 1.75z^{-1} + 1.4z^{-2} - 0.375z^{-3}}$$

$$= \frac{z^3 - z^2 - 2z}{z^3 - 1.75z^2 + 1.25z - 0.375}$$

$$= \frac{z(z-2)(z+1)}{(z-0.75)(z-0.5-j0.5)(z-0.5+j0.5)}$$

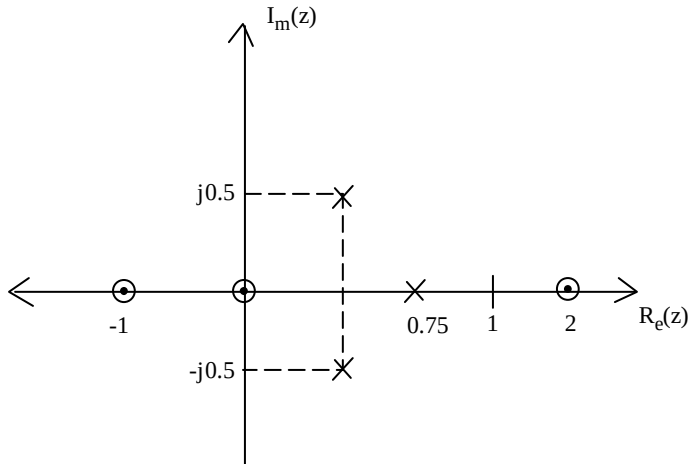


Fig: pole-zero diagram

Zeroes at z = 0, 2, -1

Poles at z = 0.75, $0.5 \pm j0.5$

A filter is characterized by following poles and zeroes on z-plane.

Zeroes at		Poles at	
Radius	Angle	Radius	Angle
0.4	π rad	0.5	0 rad
0.9	1.0376	0.892	2.5158 rad

Shows z-plane plot and plot magnitude response (not to scale).

Solution:-

Zeroes are,

$z = 0.4e^{j\pi} = -0.4$

$z = 0.9e^{j1.0376} = 0.45 + j0.77$

Poles are:

$z = 0.5e^{j0} = 0.5$

$z = 0.892e^{2.5158} = -0.728 + j0.522$

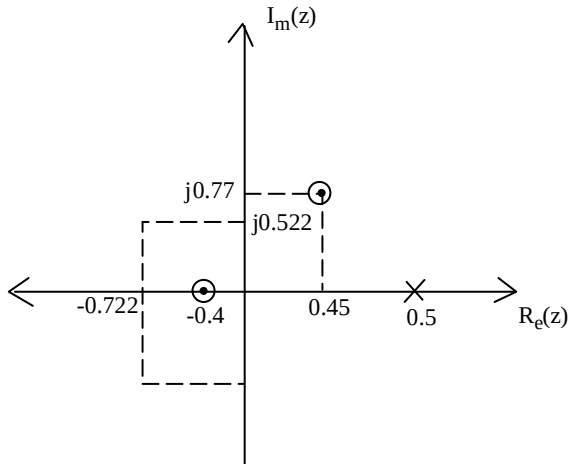


Fig: z-plane plot

$$H(z) = \frac{(z+0.4)(z-0.45-j0.77)}{(z-0.5)(z+0.77-j0.522)}$$

Put, $z = e^{jwT}$,

$$H(z) = \frac{(e^{jwT}+0.4)(e^{jwT}-0.45-j0.77)}{(e^{jwT}-0.5)(e^{jwT}+0.77-j0.522)}$$

$\left|H(e^{jwT})\right|_{WT=0}$ $\left|H(e^{jwT})\right|_{WT=3\pi/4=3.438}$

$\left|H(e^{jwT})\right|_{WT=\pi/4=0.319}$ $\left|H(e^{jwT})\right|_{WT=\pi=1.11}$

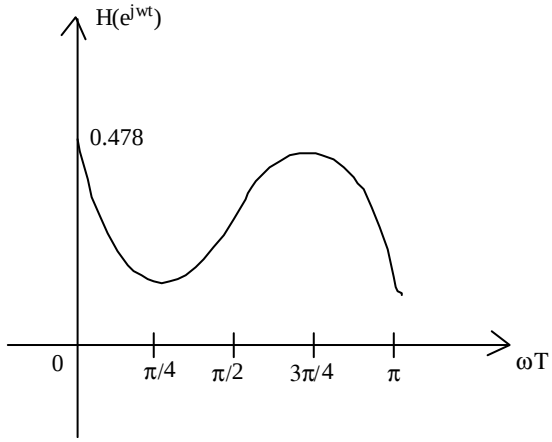
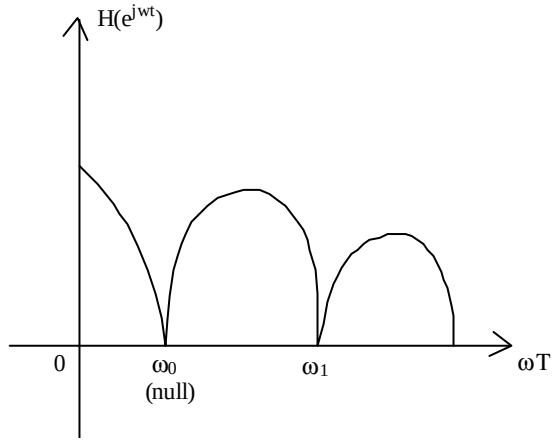


Fig: magnitude response

Notch filter:

It is a filter that contain one or more deep notches or ideally perfect nulls in its frequency response characteristics. Notch filter are useful in many application where specific frequency component must be illuminates for example, instrumentation in recording system requires that power line frequency of 50 Hz and its harmonics be illuminated.



To create a null in frequency response of filter at frequency w_0 we simply introduce a pair of complex conjugate zero on the unit circle. $z_{1,2} = e^{\pm jw_o}$

System function is given by ,

$$H(z) = b_0(1 - e^{jw_0}z^{-1})(1 - e^{jw_0}z^{-1})(1 - e^{jw_0}z^{-1})$$

Date: 2066/09/23

Response to complex exponential signal: Frequency response function:-

The input-output relationship for LTI system is $u(t) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$ ……..(i)

To develop a frequency domain characteristics of the system.

Let us excite the system with complex exponentials.

$$x(n) = Ae^{jwn} \qquad -\infty < n < \infty \quad \text{……(2)}$$

Where, A is amplitude and w is orbitrary frequency confined to the frequency interval $[-\pi, \pi]$

Now,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\left[Ae^{jw(n-k)}\right]$$

$$= \left[\sum_{k=-\infty}^{\infty} h(k)Ae^{-jwk}\right]e^{jwn} \quad \text{……..(3)}$$

$$H(w) = \sum_{k=-\infty}^{\infty} h(k)e^{-jwk} \quad \text{……(4)}$$

The function H(w) exists if the system is BIBO stable, that is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \text{……(5)}$$

The response of the system to complex exponential is given by ,

$y(n) = AH(w)e^{jwn}$ …….(6)

We note that the response is also in the form of complex exponential with the same frequency as the input but altered by multiplicative factor H(w). The multiplicative factor is called eigen value of the system in this case a complex exponential of the form (2) is an eigen function of LTI system and H(w) evaluated at frequency of input signal is corresponding eign value.

Determine the o/p sequence of the system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ When the input is complex exponential sequence $x(n) = Ae^{j\pi n/2}u(n)$ $-\infty < n < \infty$.

Solution:-

$$H(z) = \frac{1}{1 - \frac{1.}{2}z^{-1}}$$

$$H(w) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

At $w = \frac{\pi}{2}$,

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 - \frac{1}{2}e^{-j\frac{\pi}{2}}} = \frac{2}{\sqrt{5}}e^{-j26.6}$$

And therefore the o/p is

$$y(n) = A\left(\frac{2}{\sqrt{5}}e^{-j26.6}\right)e^{jn\frac{\pi}{2}}$$

$$y(n) = \frac{2}{\sqrt{5}}Ae^{j(\pi\frac{n}{2} - 26.6)}, \qquad -\infty < n < \infty$$

Determine the response of the system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ to the input signal

$x(n) = 10 - 5\sin\frac{\pi}{2}n + 2\cos\pi n \qquad -\infty < n < \infty$

Solution:-

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H(z) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

The first term in a input signal is a fixed signal component corresponding to w = 0. Thus,

$$H(0) = \frac{1}{1-\frac{1}{2}} = 2$$

The second term in x(n) has frequency $\frac{\pi}{2}$. Thus ,

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}} = \frac{2}{\sqrt{5}}e^{-j26.6}$$

Finally the third term in x(n) has a frequency $w = \pi$. Thus $H(\pi) = \frac{2}{3}$.

Hence the response of the system to x(n) is,

$$y(n) = 20 - \frac{10}{\sqrt{5}}\sin\left(\frac{\pi}{2}n - 26.6\right) + \frac{40}{3}\cos\pi n, \qquad -\infty < n < \infty$$

A LTI system is described by following difference equation $y(n) = ay(n-1) + bx(n)$
$0 < a < 1$.

(a) Determine the magnitude and phase of frequency response H(w) of the system.
(b) Choose the parameter 'b' so that the maximum value of $|H(w)|$ and phase of H(w) for a= 0.9.
(c) Determine the o/p of the system to signal for $x(n) = 5 + 12\sin\frac{\pi}{2}n - 20\cos\left(\pi n + \frac{\pi}{4}\right)$.

Date: 2066/09/25

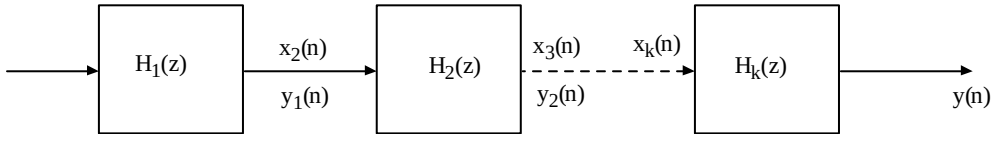
Chapter:- 5

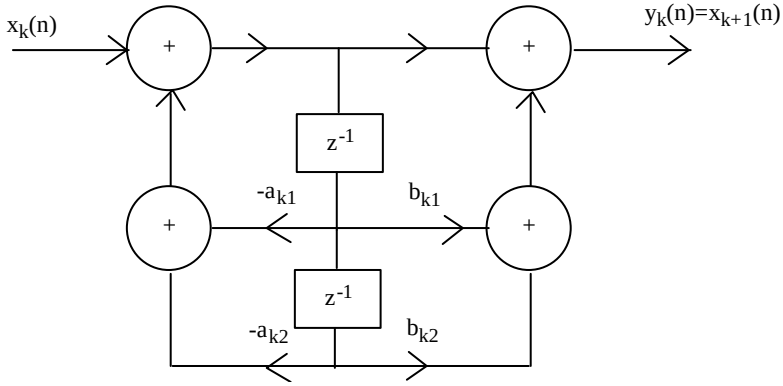
Discrete filter structure:-
Cascade form structure:-

$H(z) = \Pi_{k=1}^{k} H_k(z)$

Where k is integer part of (N+1)/2

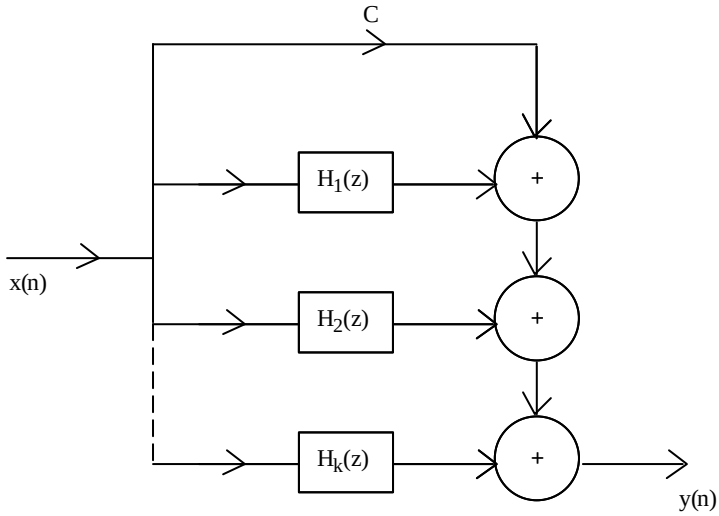
$$H_k(k) = \frac{b_{ko} + b_{k1}z^{-1} + b_{k2}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$





Parallel form structure:

$$H(z) = c + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$



Determine the cascade parallel realization for the system described by the system function

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1-2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left[1+\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

Cascade form:-

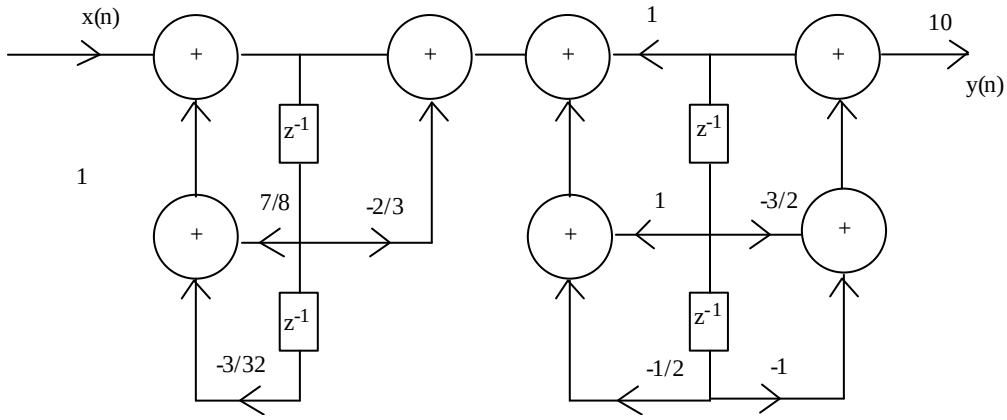
$$H_1|z| = \frac{1-\frac{2}{3}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)} = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

$$H_2(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+2z^{-1}\right)}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

$$= \frac{1+\frac{3}{2}z^{-1}-z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

$$H(z) = 10H_1(z)H_2(z)$$

The cascade relationship is



Parallel form:-

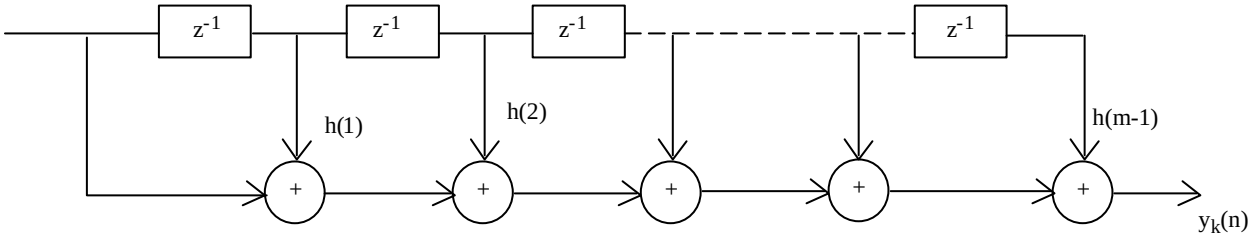
$$H(z) = \frac{10\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)(z+2)z}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)\left[z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]}$$

$$\frac{H(z)}{z} = \frac{10\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)(z+2)}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)\left[z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]}$$

$$= \frac{k_1}{z-\frac{3}{4}} + \frac{k_2}{z-\frac{1}{8}} + \frac{k_3}{z-\left(\frac{1}{2}+j\frac{1}{2}\right)} + \frac{k_3^*}{z-\left(\frac{1}{2}-j\frac{1}{2}\right)}$$

$k_1 = 2.93, \quad k_2 = -17.68$, $k_3 = 12.25 - j14.57, \qquad k_3^* = 12.25 + j14.57$

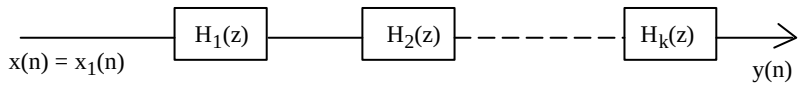
$$H(z)=\frac{2.93}{1-\frac{3}{4}z^{-1}}+\frac{-17.68}{1-\frac{1}{8}z^{-1}}+\frac{12.25+j14.57}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}}+\frac{12.25+j14.57}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

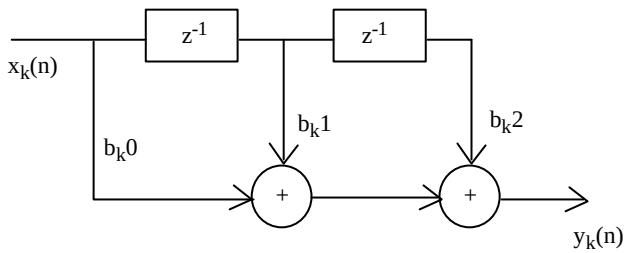


Cascade form structure:-

$H(z)=\Pi_{k=1}^{k}H_k(z)$

$H_k(z)=b_{ko}+b_{k1}z^{-1}+b_{k2}z^{-2} \qquad k=1,\ 2\ k$





Lattice form structure:

Fir filter with system function

$H_m(z)=A_m(z) \qquad m=0,1\ m-1$

$A_m(z)=1+\sum_{k=1}^{m}\alpha_m(k)z^{-k} \qquad m\geq 1$

$A_o(m)=1$

$h_m(z)=1,\quad h_m(k)=\alpha_m(k), \qquad k=1,2.....m$

$m\longrightarrow \deg ree\ \ of\ \ \ polynomial.$

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k) x(n-k)$$

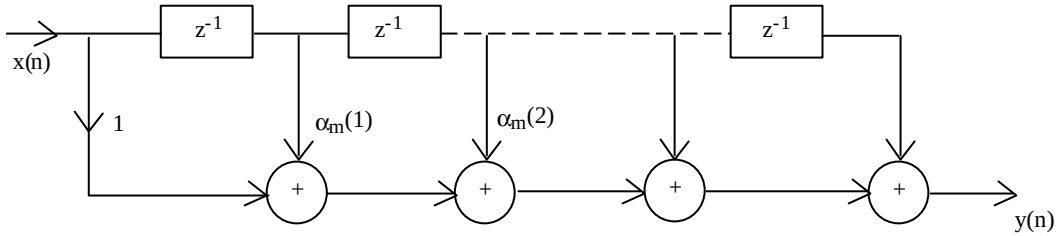
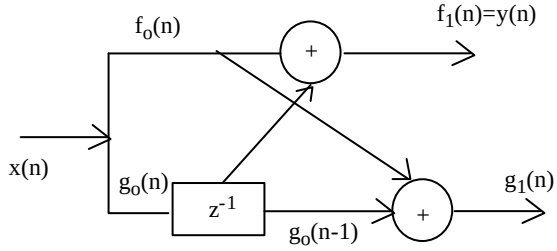


Figure: Direct form realization.

For m = 1.

$y(n) = x(n) + \alpha_1(1) x(n-1)$

1st order lattice filter.



$f_0(n) = x(n)$

$g_0(n) = x(n)$

$f_1(n) = f_0(n) + k_1 g_0(n-1)$

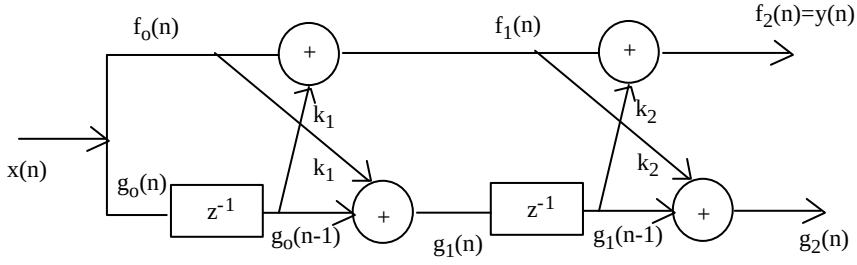
$y(n) = x(n) + k_1 x(n-1)$

$g_1(n) = k_1 f_0(n) + g_0(n-1)$

$= k_1 x(n) + x(n-1)$

$k_1 =$ *reflection coefficient*

$k_1 = \alpha_1(1)$



$y(n) = x(n) + \alpha_2(1) x(n-1) + \alpha_2(2) x(n-2)$

$f_2(n) = f_1(n) + k_2 g_1(n-1)$

$= x(m + k_1(1+k_2) x(n-1) + k_2 x(n-1)$

$= \alpha_2(2) = k_2 \quad , \quad \alpha_2(1) = k_1(1+k_2)$

Conversion of lattice coefficient to direct form filter coefficients:-

Direct form filter coefficient $\{\alpha_m(k)\}$ can be obtained form lattice coefficients $\{k_i\}$ using the relations.

$A_0(z) = B_0(z) = 1$

$A_m(z) = Am^{-1}(z) + k_m z^{-1} B_{m-1}(z) \qquad m = 1,\ 2,\ m-1$

$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = 1,\ 2,\ m-1$

$B_m(z) \rightarrow$ Reverse polynomial of $A_m(z)$.

A given 3-stage lattice filter with coefficient $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{2}$, $k_3 = \frac{1}{3}$. Determine FIR filter coefficient for direct form structure.

$A_1(z) = A_0(z) + k_1 z^{-1} B_0(z)$

$= 1 + \frac{1}{4} z^{-1}$

$\alpha_1(0) = 1, \quad \alpha_1(1) = \frac{1}{4}$, Corresponding to single stage lattice filter.

$\beta_1(z) = z^{-1} A_1(z^{-1}) = z^{-1}\left(1 + \frac{1}{4} z\right) = \frac{1}{4} + z^{-1}$

$A_2(z) = A_1(z) + k_2 z^{-1} \beta_1(z)$

$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$

$\alpha_2(0) = 1, \qquad \alpha_2(1) = \frac{3}{8}, \qquad \alpha_2(2) = \frac{1}{2}$ Corresponding to 2nd order lattice form.

$\beta_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}$

$A_3(z) = A_2(z) + k_3 z^{-1} \beta_2(z)$

$= 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$

$\alpha_3(0) = 1, \quad \alpha_3(1) = \frac{13}{24}, \quad \alpha_3(z) = \frac{5}{8}$, $\alpha_3(3) = \frac{1}{3}$

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Conversion of direct form FIR filter coefficient to lattice coefficient:

$A_{m+1}(z) = \frac{A_m(z) + k_m B_m(z)}{1 + k_m^2} \qquad m = m-1,.....-1$

$k_m = \alpha_m = \alpha_m(m)$

Determine lattice Coefficient Corresponding to FIR filter with system function.

$$H(z) = A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3} + \frac{13}{24}z^{-3} + z^{-3}$$

Solution:

$$k_1 = \alpha_1(3) = \frac{1}{3}, \qquad B_3(z) = \frac{1}{3} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}$$

$$A_2(z) = \frac{A_3(Z) - k_3 B_3(z)}{1 - k_3^2} = 1 + \frac{3}{8}z^{-1} + \frac{1}{2}z^{-2}$$

$$K_2 = \alpha_1(2) = \frac{1}{2} \quad , \quad B_2(z) = \frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}$$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - k_2^2} = 1 + \frac{1}{4}z^{-1}$$

$$k_1 = \alpha_1(1) = \frac{1}{4}$$

Lattice and lattice ladder structure for IIR system:-

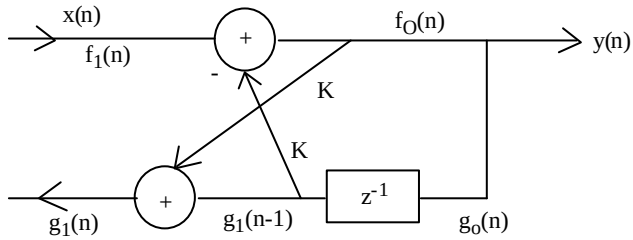
Let us begin with all-pole system with system function.

$$H(z0 = \frac{1}{1 + \sum_{k=1}^{N} a_m(k) z^{-k}} = \frac{1}{\Delta m(z)} \quad \ldots\ldots\ldots \quad (1)$$

Difference equation for this system is

$$y(k) = \sum_{k=1}^{N} a_m(k) y(n-k) + x(k) \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

When N = 1.



$x(n) = f_1(n)$

$f_o(n) = f_1(n) - k_1 g_o(n-1)$

$g_1(n) = k_1 f_0(n) + g_o(n-1)$

$y(n) = f_o(n)$

$= f_1(n) + k_1 g_o(n-1)$

$y(n) = x(n) - k_1 y(n-1)$

$y_1(n) = k_1 y(1) + y(n-1)$

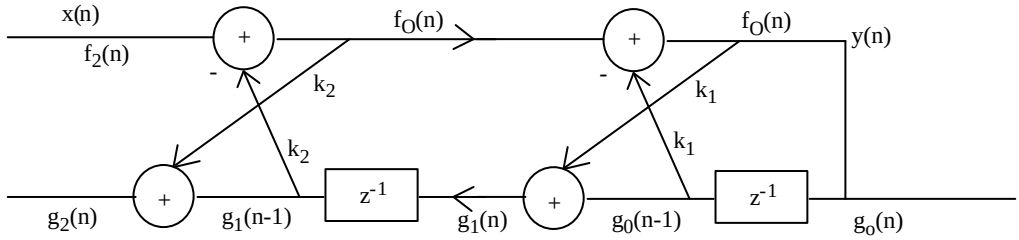


Fig: Two stage lattice structure

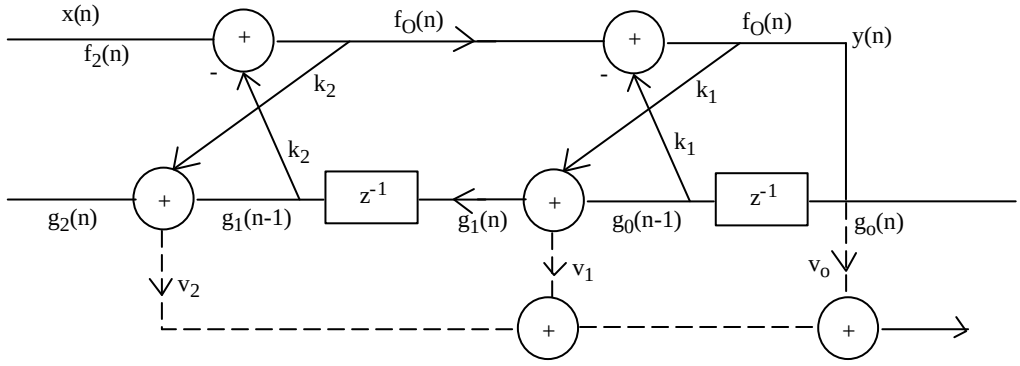
$f_o(n) = x(n)$
$f_1(n) = f_2(n) - k_2 g_1(n-1)$
$g_2(n) = k_2 f_1(n) - k_1 g_2(n-1)$
$f_o(n) = f_1(n) - k_1 g_0(n-1)$
$g_1(n) = k_1 f_o(n) + g_o(n)$
$y(n) = f_o(n) = g_o(n)$
$= f_2(n) - k_2 g_1(n-1) - k_1 g_o(n-1)$
$= -k_1 (1+k_2)\, y(n-1) - k_2\, y(n-2) + x(n)$

$$H(z) = \frac{\sum_{k=0}^{m} C_m(k) z^{-k}}{1 + \sum_{k=0}^{N} a_N(k) z^{-k}} = \frac{C_m(z)}{A_m(z)}$$

Ladder Co-efficient,

$v_m = C_m(m)$ m = 0, 1, 2, 3, ……M

$C_{m-1}(z) = c_n(Z) - v_m / B_m(z)$



Obtain the lattice ladder structure.

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}}$$
$$= \frac{C_3(z)}{A_3(z)}$$

Solution:

Lattice coefficient:-

$$A_3(z) = 1 - 0.3z^{-1} + 0.64z^{-2} - 0.576z^{-3}$$

$$k_3 = \alpha_3(3) = -0.576$$

$$B_3(z) = -0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3}$$

$$A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{1 - k_3^2} = 1 - 0.795z^{-1} + 0.1819z^{-2}$$

$$k_2 = \alpha_2(z) = 0.1819$$

$$B_2(z) = 0.1819$$

$$B_2(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - k_2^2} = 1 - 0.6726z^{-1}$$

$$k_1 = \alpha_1(1) = -0.6726$$

Ladder coefficient:-

$$C_3(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

$$v_3 = C_3(3) = 1$$

$$C_2(z) = C_3(z) - v_3 B_3(z)$$

$$= 1.576 + 2.36z^{-1} + 3.9z^{-2}$$

$$C_1(z) = C_2(z) - v_2 B_2(z)$$

$$= 0.866 + 5.46z^{-1}$$

$$v_1 = 5.46$$

$$C_0(z) = C_1(z) - v_1 B_1(z)$$

$$= 4.538$$

$$v_0 = C_0(0) = 4.538$$

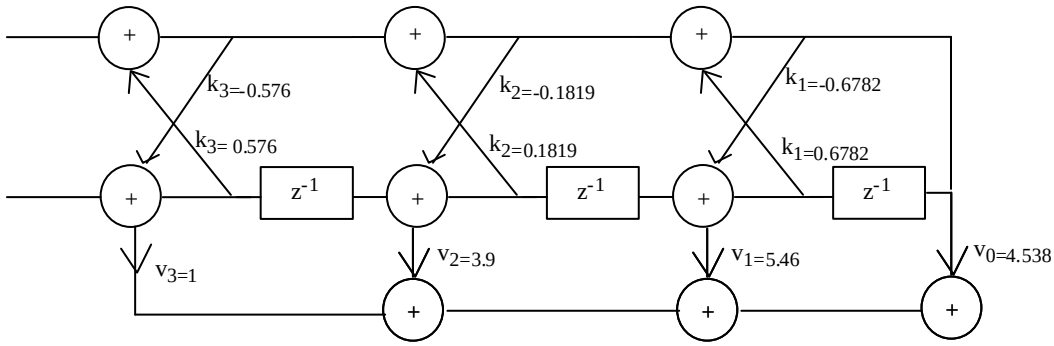


Fig: Lattice ladder structure.

Date: 2066/09/30

Analysis of sensitivity to quantization of filter coefficient:-

Let us consider a general FIR filter with system ,

$$H(z)=\frac{\sum_{k=0}^{m} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{.....(1)}$$

With quantized coefficient coefficient the system function.

$$\overline{H}(z)=\frac{\sum_{k=0}^{m} b_k z^{-k}}{1+\sum_{k=1}^{N} \overline{a}_k z^{-k}}$$

Where quantized coefficient $\{\overline{b}_k\}$ can be released to the unquatized coefficients $\{\overline{a}_k\}$ & $\{\overline{b}_k\}$ by the relation,

$$\overline{a}_k = a_k + \Delta a_k, \qquad k=1,2,3\ N \quad \text{---------(3)}$$

$$\overline{b}_k = b_k + \Delta b_k, \qquad k=1,2,3\ m \quad \text{---------} \ (4)$$

Where $\{\Delta\overline{a}_k\}$ represents the quantization errors.

The denominator of H(z) may be expressed to the form ,

$$D(z)=1+\sum_{k=1}^{N} a_k z^{-k}=\Pi_{k=1}^{N}(1-p_k z^{-1}) \ \text{.....(4)}$$

Where, $\{p_k\}$ are the poles of H(z)

Similarly we can express the denominator of $\overline{H}(z)$ as,

$$\overline{D}(z)=\Pi_{k=1}^{N}(1-\overline{p}_k z^{-1}) \ \text{.....(5)}$$

Where, $\overline{p}_k = p_k + \Delta p_k, \qquad k=1,\ 2\ N$

And Δp_k is the error or perturtation resulting form the quantization of filter coefficients.

We shall relate the perturtation errors in the $\{a_k\}$.

The perturtation error Δp_i can be expressed as,

$$\Delta p_i=\sum_{k=1}^{N}\frac{\delta p_i}{\delta a_k} \quad \text{.....} \ (6)$$

$$\frac{\delta p_i}{\delta a_k}=\frac{\delta\Delta(z)/\delta a_k}{(\delta D(z)/\delta z)_{z=p_i}} \quad \text{.....} \ (7)$$

The total perturtatio error,

$$\Delta p_i=-\sum_{k=1}^{N}\frac{p_i^{N-k}}{\Pi_{l=1,l\neq i}^{N}(p_i-p_r)}\Delta q_k \quad \text{.....} \ (8)$$

The error can be minimized by maximizing the lengths $|p_i - p_l|$. This can be accomplished by realizing the high order filter with either single pole or double pole filter sections.

Limit cycle oscillation in recursive system:-

In the realization of digital filter either in digital hardware or software on a digital computer, the quantization inherent in the finite precision arithmetic operation render the system non linear. In recursive system the non linearities due to finite precision arithmetic operation often cause periodic oscillation to occur in o/p even when the input sequence is zero or some non zero constant value. Such oscillation in recursive system are called limit cycle and are directly attributable to round off errors in multiplication and overflow error in addition.

Scaling to prevent overflow:-

In order to limit the amount of non linear distortion it is important to scale the input signal and unit sample response between the input and any internal summing mode in the system such that overflow becomes a rare event.

For fixed point arithmetic let us first consider the extra condition that overflow is not permitted at any node of the system. Let $y_k(n)$ denote the response of the system at k^{th} node when input sequence is x(n) and unit sample response between the node and the input in $h_k(z)$.

$$|y_k(n)| = \left|\sum_{m=-\infty}^{\infty} h_k(m)x(n-m)\right| \le \sum_{m=-\infty}^{\infty} |h_k(m)||x(n-m)| \quad \ldots\ldots\ldots\ldots \quad (1)$$

Suppose that x(n) is upper bounded by A_k.

Then,

$$|y_k(n)| \le A_k \sum_{m=-\infty}^{\infty} |h_k(m)| \quad \text{for all n} \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, if the dynamic range of the computer is limited to (-1, 1) the condition,

$$|y_k(n)| < 1 \quad \ldots\ldots\ldots\ldots \quad (3)$$

Can be satisfied by requiring that the input x(n) can be such that,

$$A_x < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k(m)|} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

For all possible nodes in the system. The condition in (4) is both necessary and sufficient to prevent coefficient.

For FIR filter, (4) become,

$$A_x < \frac{1}{\sum_{m=0}^{m-1} |h_k(m)|} \qquad \ldots\ldots\ldots\ldots..(5)$$

Another approach to scaling is to scale the input so that,

$$\sum_{n=-\infty}^{\infty} |y_k(n)|^2 \le C^2 \sum_{n=-\infty}^{\infty} |x(n)|^2 = C^2 E_x \ldots\ldots\ldots.. \quad (6)$$

From parsavals theorem,

$$\sum_{n=-\infty}^{\infty} |y_k(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(w)X(w)|^2$$

$$\le E_x \frac{1}{2\pi} \int |H(w)|^2 \, dw \quad \ldots\ldots\ldots\ldots.. \quad (7)$$

From (6) and (7)

$$C^2 E_x \le E_x \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(w)|^2 dw$$

$$C^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(w)|^2 dw \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Read the following topics your self:-

- Representation of Number.
 - Fixed point representation.
 - Floating point representation.
- Rounding and truncation error.

Chapter: 6

Design of IIR filter form analog filters:-

Impulse invariance method:

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s - p_i}$$

Taking inverse laplace transform

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t)$$

Put t = nT

$$h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT)$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT)\right] z^{-1}$$

$$= \sum_{i=1}^{M}\sum_{n=0}^{\infty} A_i \left(e^{p_i T} z^{-1}\right)^n u_a(nT)$$

$$= = \sum_{i=1}^{M} A_i \frac{1}{1 - e^{p_i T} z^{-1}}$$

Hence,

$$\frac{1}{s - p_i} \rightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$$

Q. Convert the analog filter with the system function $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into a digital IIR by means of impulse invariance method.

Solution:

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{s+0.1}{(s+0.1+j3)(s+0.1-j3)}$$

$$= \frac{k_1}{(s+0.1+j3)} + \frac{k_1^*}{(s+0.1-j3)}$$

$$k_1 = \frac{1}{2}, \quad k_1^* = \frac{1}{2}$$

$$H_a(s) = \frac{\frac{1}{2}}{(s+0.1+j3)} + \frac{\frac{1}{2}}{(s+0.1-j3)}$$

Then using impulse invariance method,

$$H(z) = \frac{\frac{1}{2}}{1+e^{(-0.1-j3)T}z^{-1}} + \frac{\frac{1}{2}}{1-e^{(-0.1-j3)T}z^{-1}}$$

$$= \frac{1-\left(e^{-0.1T}\cos 3T\right)z^{-1}}{1-\left(ze^{-0.1T}\cos 3T\right)z^{-1}+e^{-0.2T}z^{-1}}$$

IIR filter design by bilinear transformation:

IIR filter deign using impulse invariance and approximation of derivative method have a sever limitation in that they are approximate only for low pass filter and limited calls of bandpass filter.

In this we describe a mapping from s-plane to z-plane called bilinear transformation that overcomes the limitation of other two design method describe previously.

Let us consider the analog filter with system function.

$$H(s) = \frac{b}{s+a} \quad \text{..... (1)}$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

Or, $sY(s) + aY(s) = bX(s)$

Taking inverse laplace transform, we get,

$$\frac{du(t)}{dt} + ay(t) = bx(t)$$

Taking integration on both sides.

$$\int \frac{du(t)}{dt}dt + a\int_{m-T}^{nT} y(t)dt = \int_{nT-T}^{nT} x(t)dt$$

Using Trapezoidal rule,

$$\int_{nT-T}^{nT} g(t)dt = T/2\left[g(nT) + g(nT-T)\right]$$

Then,

$$y(nT) - y(nT - T) + aT/2\left[y(nT) + y(nT - T)\right] = bT/2\left[x(nT) + x(nT - T))\right]$$

Taking z-transform on both sides,

$$Y(z) - z^{-1}Y(z) + aT/2\left[Y(z) + z^{-1}Y(z)\right] = bT/2\left[X(z) + z^{-1}X(z)\right]$$

$$\frac{Y(z)}{X(z)} = \frac{bT/2\left(1 + z^{-1}\right)}{1 - z^{-1} + aT/2\left(1 + z^{-1}\right)}$$

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a}$$

Hence, $s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$

It gives the mapping form s-plane to z-plane. This is called bilinear transformation.

To investigate the characteristics of bilinear transformation let,

$$z = re^{jw}$$

$$s = \sigma + j\Omega$$

Now , $s = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right)$

$$= \left(\frac{re^{jw} - 1}{re^{jw} + 1}\right)$$

$$= \frac{2}{T}\left(\frac{r^2 - 1 + j2r\sin w}{r^2 + 2r\cos w + 1}\right)$$

Comparing with,

$$s = \sigma + j\Omega$$

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{r^2 + 2r\cos w + 1}\right)$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin w}{r^2 + 2r\cos w + 1}\right)$$

If r <1, then $\sigma < 0$ and if r >1, then, $\sigma > 0$ consequently, the LHP is S maps into inside a unit circle in z-plane and RHP in s maps into outside a unit circle.

When, r = 1, $\sigma = 0$,

$$\Omega = \frac{2}{T}\left(\frac{2\sin w}{2 + 2\cos w}\right)$$

$$\Omega = \frac{2}{T}\tan(w/2)$$

$$w = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

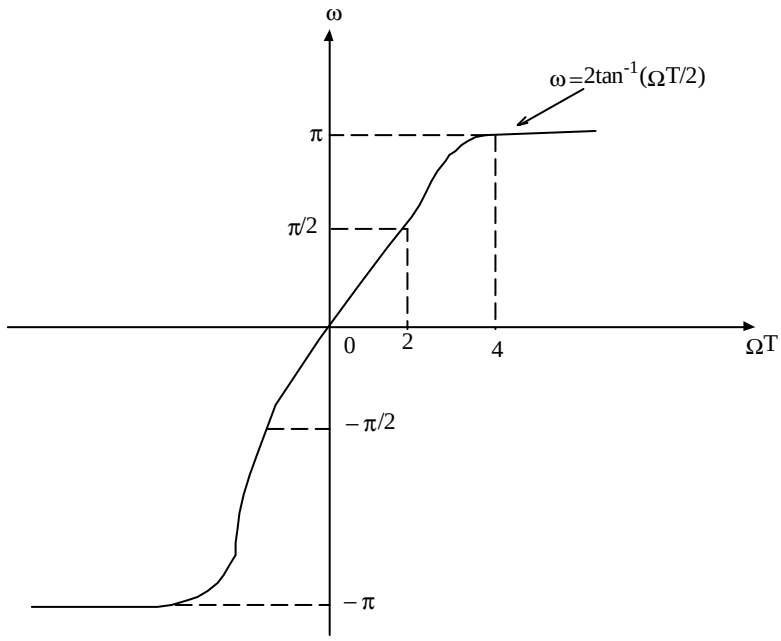


Fig: mapping between frequency variables w and Ω resulting from bilinear transformation.

Convert the analog filter with system $H_a(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$ *into digital IIR filter by means of bilinear transformation the digital is to have resonant frequency of* $w_r = \pi/2$.

Solution:

The resonant frequency of analog filter is $\Omega_r = 4$. & resonant frequency of digital filter is

$$\Omega_r = \frac{2}{T}\tan w_r/2$$

$$4 = \frac{2}{T}\tan\frac{\pi}{4}$$

$$\therefore T = \frac{1}{2}$$

Now , $s = 2/T\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$

Using bilinear, the system function of digital filter becomes $H(z) = \dfrac{4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right]^2+16}$

$$= \frac{0.125+0.006z^{-1}-0.118z^{-2}}{1+0.0006z^{-1}+0.95z^{-1}}$$

$$H(z) = \frac{0.125+0.006z^{-1}-0.118z^{-2}}{1+0.95z^{-2}}$$

Q. Design a single pole low pass digital filter factor with 3 dB bandwidth of 0.2 π using the bilinear transformation apply to the analog filter $H(s) = \frac{\Omega_c}{s + \Omega_c}$. *Where,* Ω_c *is a 3dB bandwidth analog filter.*

Solution:

$w_c = 0.2\pi$,

For analog filter.

$$\Omega_c = \frac{2}{T}\tan w_c / 2$$

$$= \frac{2}{T}\tan(0.1\Omega(\))$$

$$= \frac{0.65}{T}$$

Analog filter has system function

$$H(s) = \frac{0.65/T}{s + 0.6/T}$$

Now, $s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$

Then,

$$H(z) = \frac{0.65/T}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + \frac{0.65T}{T}}$$

$$= \frac{0{,}65(1 + z^{-1})}{2.65 - 1.35z^{-1}}$$

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

Matched z-transformation:-

Another method for converting analog filter into equivalent digital filter is to map the poles and zeroes of H(s) directly into poles and zero in z-plane.

Suppose the transfer function of analog filter is expressed in the factored formed.

$$H(s) = \frac{\Pi_{k=1}^{M}(s - z_k)}{\Pi_{k=1}^{N}(s - p_k)}$$

Where, $\{z_k\}$ and $\{p_k\}$ are zeroes and poles of the analog filter.

Then system function for digital filter is

$$H(s) = \frac{\Pi_{k=1}^{M}(1 - e^{z_k T} z^{-1})}{\Pi_{k=1}^{N}(1 - e^{p_k T} z^{-1})}$$

Thus each factor of the form (s-a) in H(s) is mapped into the factor $(1 - e^{aT} z^{-1})$. This mapping is called matched z-transformation.

Convert the analog filter with system function $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into digital IIR filter by matched z-transformation method.

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2 - (3j)^2} = \frac{(s+0.1)}{(s+0.1+3j)(s+0.1-j3)}$$

Using matched z-transformation method.

$$H(z) = \frac{\left(1-e^{-0.1T}z^{-1}\right)}{\left(1-e^{(-0.1-3j)T}z^{-1}\right)\left(1-e^{(-0.1+j3)T}z^{-1}\right)}$$

$$= \frac{(1-e^{0.1T}z^{-1})}{(1-e^{-0.1T}e^{-j3T}z^{-1})(1-e^{-0.1T}e^{j3T}z^{-1})}$$

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Butter worth filter:

We have, $|T_n(jw)|^2 = \frac{1}{1+\varepsilon^2\omega^{2n}}$

When $\varepsilon^2 = 1$

$$|T_n(jw)|^2 = \frac{1}{1+\omega^{2n}}$$

This function is known as Butterworth response. From this equation we observe some interesting properties of Butterworth response,

(1) The Butterworth filter is an all –pole filter. It has zero at infinity $(\omega \to \infty)$.

(2) $|T_n(j0)| = 1$ for all n.

(3) $|T_n(j1)| = \frac{1}{\sqrt{2}} = 0.707$ for all n. corresponding to 3 dB.

(4) For large ω, $|T_n(j\omega)|$ exhibits n-pole roll-off.

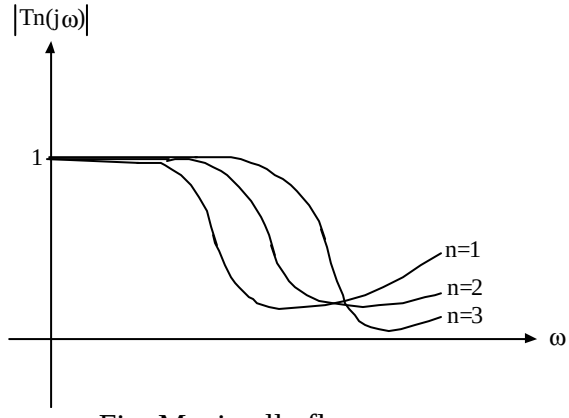


Fig: Maximally flat response.

Butterwoth filter:-

$$\alpha(\omega) = -20\log|T(jw)|dB$$

Low pass filter specifications,

Passband frequency = Ω_p

Stopband frequency = Ω_s

Passband attenuation = α_p

Stopband attenuation = α_s

Order of filter,

$$n = \frac{\log\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}{2\log\left(\Omega_s/\Omega_p\right)}$$

3-dB cutoff frequency,

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{\frac{1}{2n}}}$$

Chebyshev filter:

We have chebyshev magnitude response,

$$|T_n(jw)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$C_n(\omega) = \cos(n\cos^{-1}\omega) \quad \text{for} \quad |\omega \le 1|$$

$$= \cosh(n\cosh^{-1}\omega) \quad \text{for} \quad |\omega| > 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

With equation (2), the transfer function magnitude in (1) is determined for all values of ω. The function is plotted for n = 6.

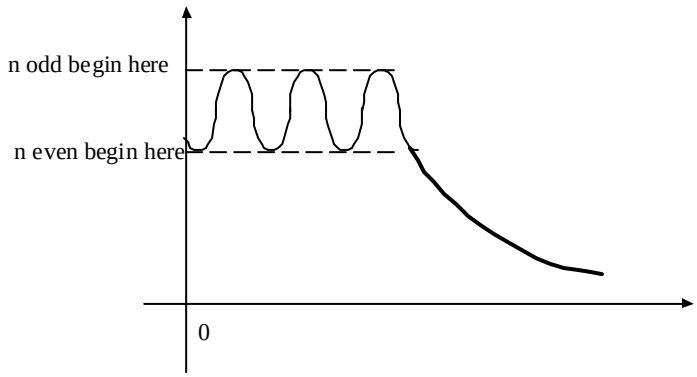


Fig: Sixth order chebyshev T_F magnitude

(1) $C_n(0) = 0$ for odd and $C_n(0) = 1$ for n even.

$|T_n(j0)| = 1$ for n odd.

$$|T_n(j0)| = \frac{1}{\sqrt{1+\varepsilon^2}} \quad \text{for n even.}$$

(2) At $\omega = 1$, $C_n(1) = 1$

$$|T_n(j0)| = \frac{1}{\sqrt{1+\varepsilon^2}} \quad \text{for all n.}$$

$$n = \frac{\cosh^{-1}\left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)}{\cosh^{-1}\left(\Omega_s / \Omega_p\right)}$$

- equal ripple filter.
- High attenuation in stop band and steeper roll-off near the cut-off frequency.

* For $\Omega_p = 1,\ \Omega_s = 3.33$ *and* $\alpha_p = 0.3dB\ \ \alpha_s = 22dB$ Which filter is best.

Solution,

For butterworth filter n = 5. for chebyshev filter n = 3.

Greater efficiency of chebyshev filter compared with Butterworth filter.

Frequency transformation:-

If we wish to design a high pass or bandpass or bandstop filter it is a simple method to take a low pass prototype filter (butterwoth , chebyshev) perform a frequency transformation.

One possibility is to perform the frequency transformation in analog domain and then to convert analog filter into corresponding digital filter by a mapping of s-plane into z-plane and alternative approach is first to convert the analog low pass filter into digital low pass filter into a desired digital filter by a digital transformation.

In general, these two approaches yields different results except for bilinear transformation in which case the resulting filter designs are identical.

Frequency transformation in analog domain:-

Type of transformation	Transformation	Band edge freq. of new filter.
Low pass	$S \to \frac{\Omega_p}{\Omega_{p'}} s$	$\Omega_{p'}$
High pass	$S \to \frac{\Omega_p \Omega_{p'}}{s}$	$\Omega_{p'}$
Band pass	$S \to \Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}$	Ω_L , Ω_u
Band stop	$S \to \Omega_p \frac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_L \Omega_u}$	Ω_L , Ω_u

Q. Transform the single pole low pass butterworth filter with the system function $H(s) = \frac{\Omega_p}{s + \Omega_p}$ into band pass filter with upper and lower band edge frequency Ω_u and Ω_L respectively.

Solution:

$$S = \Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}$$

$$H(s) = \frac{\Omega_p}{\Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} + \Omega_p}$$

$$= \frac{s(\Omega_u - \Omega_L)}{s^2 + (\Omega_u - \Omega_L)s + \Omega_L \Omega_u}$$

Frequency transformation in digital domain (spectra transform):

Conversion of prototype digital low pass filter into either band pass, band stop, high pass, or another low pass digital filter.

Need of spectral transformation:-

Impulse invariance method and mapping of derivative are inappropriate to use in designing high pass and many band pass filter due to a aliasing problem.

Consequently one could not employ an analog frequency transformation followed by conversion of the result into digital domain by use of this two mapping instead it is much better to perform the mapping analog low pass filter into digital low pass filter by either of these mapping and then to perform frequency transformation in digital domain.

The problem of aliasing is avoided.

Type of transformation	**Transformation**	**parameter**
Low pass	$z^{-1} \rightarrow \frac{z^{-1} - a}{1 - az^{-1}}$	$\omega_p^{'} = band\ edge\ frequency\ of\ new\ filter$ $a = \frac{\sin\left[\left(\frac{\omega_p - \omega_{p[}^{'}}{2}\right)\right]}{\sin\left[\left(\frac{\omega_p + \omega_p^{'}}{2}\right)\right]}$
High pass	$z^{-1} \rightarrow -\frac{z^{-1} + a}{1 + az^{-1}}$	$a = -\frac{\cos\left[\left(\frac{\omega_p + \omega_{p[}^{'}}{2}\right)\right]}{\cos\left[\left(\frac{\omega_p - \omega_p^{'}}{2}\right)\right]}$
Band pass	$z^{-1} \rightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$\omega_L = Lower\ band\ edge\ frequency$ $\omega_u = Upper\ band\ edge\ frequency$ $a_1 = \frac{2\alpha k}{k+1}$ $a_2 = \frac{k-1}{k+1}$

		$a = -\dfrac{\cos\left[\left(\dfrac{\omega_u + \omega_L^{'}}{2}\right)\right]}{\cos\left[\left(\dfrac{\omega_u - \omega_L^{'}}{2}\right)\right]}$ $k = \cot\left(\dfrac{\omega_u + \omega_L^{'}}{2}\right).\tan(\omega_p / 2)$
Band stop	$z^{-1} \rightarrow \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$a_1 = -\dfrac{2\alpha}{k+1},\ a_2 = \dfrac{1-k}{1+k}$ $a = \dfrac{\cos\left[\left(\dfrac{\omega_u + \omega_L^{'}}{2}\right)\right]}{\cos\left[\left(\dfrac{\omega_u - \omega_L^{'}}{2}\right)\right]}$ $k = \tan\left(\dfrac{\omega_u - \omega_L^{'}}{2}\right).\tan(\omega_p / 2)$

Design a low pass butterworth filter to meet the following specifications:-
Passband gain = 0.89
Passband frequency = 30 hz.
Attenuation = 0.20
Stopband edge = 75 Hz.

Solution:

$\Omega_p = 2\pi \times 30 = 188.4\ rad/\sec$

$\Omega_s = 2\pi \times 75 = 471.2\ rad/\sec$

$\alpha_p = -20\log(0.89) = 1.01dB$

$\alpha_s = -20\log(0.20) = 13{,}98dB$

$$n = \frac{\log\left(\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)}{2\log\left(\Omega_p / \Omega_s\right)} = 2.466 \approx 3$$

3 –dB cut off frequency,

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)} = 23.55\ rad/\sec$$

From table for N = 3, and Ω_c =1, we have

$$H(s) = \frac{1}{(s+1)/(s^2 + s + 1)}$$

This function is normalized for Ω_c =1. However Ω_c = 235.55, we need to denormalized H(s) by Ω_c =235.55 rad/sec.

$$H(s)\Big|_{s=s/\Omega_L} = \frac{1}{(s/\Omega_c+1)(s^2/\Omega_c^2+s/\Omega_c+1)}$$

Where, Ω_c = 235.55 rad/sec.

Design a low pass FIR filter with specification $\omega_p = 0.2\pi$, $\omega_s = 0.65\pi$, $\alpha_p = 0.4dB$, $\alpha_s = 1.5dB$. Use bilinear transformation method.

Solution:

$$\alpha_p = \frac{2}{T}\tan(\omega_p/2) = \frac{2}{T}\tan(0.2\pi/2) = 0.649(T=1))$$

$$\Omega_s = \frac{2}{T}\tan(\omega_s/2) = 3.263$$

Order of filter,

$$n = \frac{\log\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}{2\log(\Omega_p/\Omega_s)} = 1.783 \approx 2$$

3 –dB cut off frequency,

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{\frac{1}{2n}}} = 1.164$$

From table for n = 2 and $\Omega_c = 1$, we have

$$H(s) = \frac{1}{s^2+1.4145+1}$$

Now denormalizing with $\Omega_c = 1.164$

$$H(s)\Big|_{s=s/\Omega_c} = \frac{1}{s^2/\Omega^2+1.41s/\Omega_c+1}$$

$$= \frac{\Omega_c^2}{s^2+1.414s\Omega_c+\Omega_c^2}$$

$$= \frac{\Omega}{s^2+1.414\ s\Omega_c+\Omega_c^2}$$

Now using bilinear transformation,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = 2\left(\frac{z-1}{z+1}\right), (T=1)$$

$$H(z) = \frac{1.354}{\left[2\left(\frac{z-1}{z+1}\right)\right]^2+1.645\left[2\left(\frac{z-1}{z+1}\right)\right]+1.3548}$$

Q. Convert a given $H_p(s)=\frac{0.2}{s+0.2}$ of band edge frequency $\Omega_p=0.2$ into a high pass filter $H_n(s)$ with pass band edge frequency $\Omega_p^{'}=0.5$

Solution:

$$S\rightarrow\frac{\Omega_p\Omega_p^{'}}{s}=\frac{0.2\times0.5}{s}=\frac{0.1}{s}$$

$$H_n(s)=H_p\left(\frac{\Omega_p\Omega_p^{'}}{s}\right)=\frac{0.2}{0.1/s+0.2}=\frac{0.25}{0.25+0.1}$$

$$H_n(s)=\frac{s}{s+0.5}$$

Q. Use bilinear transformation to obtained digital low pass filter to approximate $H(s)=\frac{1}{s^2+\sqrt{2}s+1}$.

Assume cut off frequency of 100 Hz and sampling frequency of 1 khz.

Solution:

$$\omega_c=\frac{2\pi\times100}{1000}\ \left(Normalizing\ with\ f_s\right)$$

$$=0.2\pi\ \text{rad/sec}$$

$$\Omega_c=\frac{2}{T}\tan(\omega_c/2)=0.65\ (T=1)$$

Now, denormalizing H(s) with $\Omega_c=0.65$ we get,

$$H_L(s)=H(s)\Big|_{s=s/\Omega_c}=\frac{1}{\left(\frac{s}{0.65}\right)^2+\sqrt{2}(s/0.65)+1}$$

$$=\frac{0.4225}{s^2+0.919s+0.4225}$$

Now we substitute, $s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)=2\left(\frac{z-1}{z+1}\right)(T=1)$

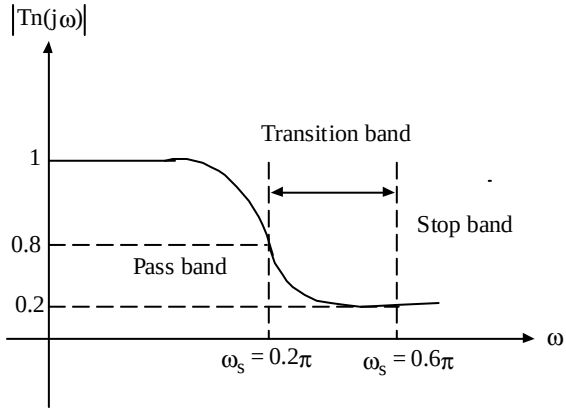
$$H(s)=\frac{0.4225}{\left[2\left(\frac{z-1}{z+1}\right)\right]^2+0.919\left[2\left(\frac{z-1}{z+1}\right)\right]+0.4225}$$

Q. Using bilinear transformation design a butterwoth filter which satisfy the following condition.

$0.8\le\left|H(e^{jw})\right|\le1$ $\qquad 0\le\omega\le0.25$

$\left|H(e^{jw})\right|\le0.2$ $\qquad 0.6\le\omega\le\pi$

Solution:



$$\alpha_p = -20\log(0.8) = 0.93dB$$

$$\alpha_s = -20\log(0.2) = 13.97dB$$

$$\omega_p = 0.2\pi \quad , \quad \omega_s = 0.6\pi$$

$$\alpha_p = \frac{2}{T}\tan(\omega_p/2) = 0.65$$

$$\alpha_s = \frac{2}{T}\tan(\omega_s/2) = 0.75$$

$$n = \frac{\log\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}{2\log(\Omega_p/\Omega_s)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{\frac{1}{2n}}} = 0.75$$

From table, for n = 2 and $\Omega_c = 1$

$$H(s) = \frac{1}{s^2+1.414s+1}$$

Now, denormalizing with $\Omega_c = 0.75$

$$H(s)\Big|_{s=s/\Omega_c} = \frac{1}{(s/0.75)^2+1.414(s/0.75)+1}$$

$$= \frac{0.56}{s^2+1.065+0.56}$$

Using bilinear transformation,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \ (r=1))$$

$$H(s) = \frac{0.56}{\left[2\left(\frac{z-1}{z+1}\right)\right]^2 + 1.06\left[2\left(\frac{z-1}{z+1}\right)\right] + 0.56}$$

Q. Convert low pass butterworth filter with system function $H(s)=\frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$ into bandpass filter with upper and lower cut off frequency ω_u and ω_c respectively. The low pass filter has 3 dB bandwidth $\omega_p = 0.2\pi.$ $(\omega_u = 3\pi/5,\ \omega_L = 2\pi/5)$

Solution:

$$k = \cot\left(\frac{\omega_u - \omega_L}{2}\right)\tan(\omega_p/2)$$

$$= 1$$

$$\alpha = \frac{\cos\left[\frac{\omega_u + \omega_L}{2}\right]}{\cos\left[\frac{\omega_u - \omega_L}{2}\right]} = 0$$

$$a_1 = \frac{-2\alpha_k}{k+1} = 0$$

$$a_2 = \frac{k-1}{k+1} = 0$$

Now, $z^{-1} \to -z^{-2}$

TF becomes,

$$H(z) = \frac{0.245(1-z^{-2})}{1+0.509z^{-2}}$$

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Chapter: 7

FIR filter design:-

FIR filter:- The digital filters can be classified either as finite duration unit impulse response (FIR) filters or infinite duration unit impulse response (IIR) filters depending upon the form of unit impulse response of the system. In FIR system the impulse response sequence is of finite duration.

For example the system with impulse response,

$$h(z) = \begin{cases} 2 & for\ |n| \le\ is\ FIR\ \ system \\ 0 & otherwise \end{cases}$$

FIR filters are generally implemented using structures with no feedback (i.e non recursive structure).

An FIR filters of length 'N' can be describe by the following difference equation.

$$y(n) = b_0 x(n) + b_1 x(n-1) + ...b_{n-1} x(n-m+1)$$

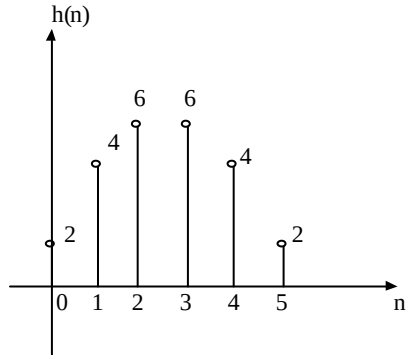
$$or, y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$ Where, b_k is filter coefficient.

Symmetric and antisymmetric FIR filters:

The unit sample response of FIR filter is symmetric if it satisfies the following condition.

$h(n) = h(M-1-n)$, n = 0, 1, 2, …… M-1.

As in example let us consider the unit sample response given in figure for M = 6 sample.



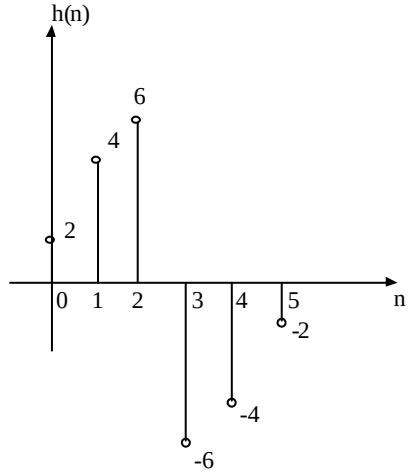
h(n) = h (5-n)
h(0) = h(5) =2
h(1) = h(4) = 4
h(2) = h(3) = 6

This is symmetric FIR filter.

The unit sample response of FIR filter is antisymmetic if it satisfies the following condition.

h(n) = -h (m-1-n) , n = 0, 1,……M-1

As an example let us consider the unit sample response shown in figure for M = 6 samples.



h(n) = -h(5-n)
h(0)=-h(5) =2
h(1)=-h(4)=4
h(2)=-h(3)=6
This is antisymmetric FIR filter.

Note: The condition for linearity phase FIR filters is h(n) $= \pm h(M-1-n)$

The phase of linear phase filter is $e^{-j\omega\left(\frac{M-1}{2}\right)}$

Show that the digital FIR filter with impulse response h(n).

$h(n) = \{2,4,6,6,4,2\}$ is liner phase system. Is this antisymmetric?

Solution:

$h(n) = \pm h(M-1-n)$

Here, M = 6.

$h(n) = \pm h(5-n)$

Where,

h(0) = 2, h(1) = 4, h(2) = 6, h(3) = 6

h (4) = 4, h(5) = 2,

h(0) = h(5) = 2

h(1) = h(4) = 4

h(2) = h(3) = 6

Hence the system has linear phase and h(n) is not antisymmetric since h(n) = h(M-1-n)

A digital filter has impulse response given by h(n) = { 1, 0, 0, 0, 0, 0, -1}. What is its system function. Which class of linear phase filter does this system belong to ? Justify.

Solution:

System function,

$$H(z) = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= 1 - z^{-6}$$

Linear phase FIR filter satisfies the conditions,

$$h(n) = \pm h(M-1-n)$$

Here, M= 7, h(n) = ± h(6-n)

h(0) = -h(6) = 1

h(1) = - h(5) = 0

h(2) = -h(4) = 0

h(3) = -h(3) = 0

Hence, this system belong to antisymmetric.

A linear phase filter has a phase function e^{-j2w}. What is the order of the filter.

Solution:

The phase of linear phase filter is given by,

$$= e^{-j\omega\left(\frac{M-1}{2}\right)}$$

Comparing with e^{j2w}

$$\frac{M-1}{2} = 2 \Rightarrow M = 5$$

Design of FIR filter using fourier series method:-

The frequency response $H_d(e^{jw})$ of a system is periodic in 2π .We know that , periodic function can be expressed as linear combination of complex exponentials. Therefore the desire frequency response of FIR filter may be represented by fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega_n} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Where, the fourier coefficients $h_d(n)$ are the impulse response sequence of the filter given by,

$$h_d(n) = \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \quad \ldots\ldots\ldots\ldots \quad (2)$$

Also, z-transform of the sequence is given by,

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n) z^{-n} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Equation (3) represents a non-causal digital filter of infinite duration to get an FIR filter transfer function the series can be truncated by assigning.

$$h(n) = \begin{cases} h_d(n) & for \ \ |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases} \quad \ldots\ldots\ldots\ldots \quad (4)$$

Then we have,

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-1} \quad \ldots\ldots\ldots\ldots\ldots..(5)$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + h(1)z^{-1} + \ldots\ldots\ldots h(0) + \ldots\ldots + h\left[-\left(\frac{N-1}{2}\right)\right] z^{(N-1/2)}$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)z^{-n} + h(-n)z^{n}\right] \quad \ldots\ldots\ldots\ldots \quad (6)$$

For symmetrical impulse response symmetrical at n = 0.

h(-n) = h(n) …… (7)

Therefore equation (6) becomes,

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right] \quad \ldots\ldots\ldots\ldots.. \quad (8)$$

This TF is not physically realizable. Realizability can be brought by multiplying the equation (8) by $z^{\left(N-\frac{1}{2}\right)}$.

Where, $\dfrac{N-1}{2}$ is delay in samples.

We have,

$$H'(z) = z^{-\left(N-\frac{1}{2}\right)H(z)}$$

$$= z^{-\left(N-\frac{1}{2}\right)}\left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right]\right] \quad \ldots\ldots\ldots\ldots.. \quad (9)$$

Design ideal lowpass filter with following frequency response.

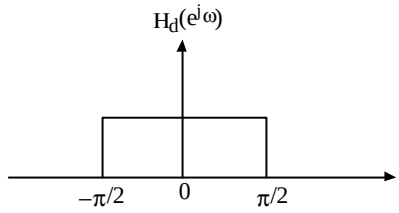
$$H_d(e^{j\omega}) = \begin{cases} 1 & for -\pi/2 \le \omega \le \pi/2 \\ 0 & for \ \pi/2 \le \omega \le \pi \end{cases}$$

Determine the values of h(n) for N = 11. Find frequency response.

Solution:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega n}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1.e^{j\omega n} d\omega$$



$$= \frac{\sin n\frac{\pi}{2}}{n\pi} \quad , \quad -\infty < n < \infty$$

Now truncating $h_d(w)$ to 11 samples, we get,

$$h(n) = \begin{cases} \dfrac{\sin n\frac{\pi}{2}}{\pi n} & for\,|n \le 5| \\ 0 & otherwose \end{cases}$$

h(0) = ½
h(1) = h(-1) = 0.3183
h(2) = h(-2) = 0
h(3) = h(-3) = -0.106
h(4) = h(-4) = 0
h(5) = h(-5) = 0.06366

TF, H(z) $= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^n + z^{-n}\right]$

$$= 0.5 + \sum_{n=1}^{5} h(n)\left[z^n + z^{-n}\right]$$

$$= 0.5 + 0.3183(z^1 + z^{-1}) - 0.106(z^3 + z^{-3}) + 0.06366\left(z^5 + z^{-5}\right)$$

TF, of realizable filter will be,

$$H'(z) = z^{-\left(N+\frac{1}{2}\right)} H(z) = z^{-5} H(z)$$

$$= 0.06366 - 0.106z^{-2} + 0.318z^{-4} + 0.5z^{-5} + 0.318z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

From above, the filter coefficients of causal filer are given as,

h(0) = h(10) = 0.06366
h(1) = h(9) = 0
h(2) = h(8) = -0.106
h(3) = h(7) = 0
h(4) = h(6) = 0.013183
h(5) = 0.5

Window techniques for designing FIR filters:

In this method we begin with the design frequency response specification $H_d(w)$ and determine the corresponding unit sample $H_d(n)$ indeed $H_d(n)$ is related to $H_d(w)$ by the fourier transform relation.

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n} \quad \ldots\ldots\ldots \quad (1)$$

Where,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \quad \ldots\ldots\ldots\ldots \quad (2)$$

In general unit sample response $H_d(n)$ obtained form equation (2) is infinite in duration and must be truncked at some point say at n = M-1, to yield FIR filter of length 'L'. Truncation of $H_d(n)$ to a length M-1 is equivalent to multiplying $H_d(n)$ rectangular window defined as,

$$\omega(w) = \begin{cases} 1 & n = 0,1,2.....M-1 \\ 0 & otherwise \end{cases} \quad \ldots\ldots\ldots\ldots \quad (3)$$

This unit sample response of FIR filter becomes ,

$$\text{h(n)} = h_d(n)\omega(n) = \begin{cases} h_d & n = 0,1.....M-1 \\ 0 & otherwise \end{cases} \quad \ldots\ldots\ldots \quad (4)$$

It is instructive to consider the effect of window function on the desire frequency response $H_d(\omega)$. The multiplication of the window function $\omega(w)$ with $h_d(n)$ is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$. Where $W(\omega)$ is the frequency domain representation of window function.

$$W(\omega) = \sum_{n=0}^{M-1} W(n) e^{-j\omega n} \quad \ldots\ldots\ldots \quad (5)$$

Thus the combination of $H_d(\omega)$ with $W(\omega)$ yields the frequency response of FIR filter.

$$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(v) W(\omega - v) dv$$

Fourier transform of rectangular window is ,

$$\begin{aligned} W(\omega) &= \sum_{n=0}^{M-1} e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} \\ &= e^{-j\omega(M-1/2)} \frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned}$$

The window function has a magnitude response,

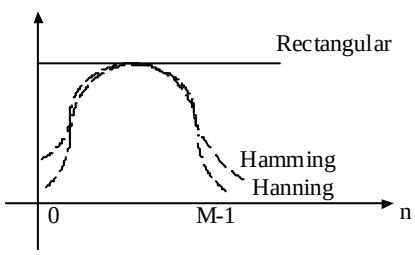
$$|W(\omega)| = \frac{|\sin(\omega M/2|}{|\sin(\omega/2)|} \qquad -\pi \le \omega \le \pi$$

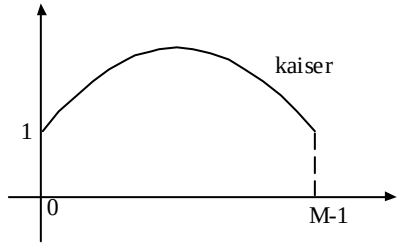
And a piece wise linaer phase .

$$\begin{aligned} \Theta(\omega) &= \omega\left(\frac{M-1}{2}\right) \quad \text{When,} \quad \sin(\omega M/2) \ge 0 \\ &= -\omega\left(\frac{M-1}{2}\right) + \pi \quad \text{When,} \quad \sin(\omega M/2) < 0 \end{aligned}$$

Date: 2066/10/11

Name of window	Time domain sequence $\omega(n),\ 0 \le n \le -1$
Rectangular	1
Hamming	$0.54\text{-}0.46\cos\frac{2\pi n}{M-1}$
Hanning	$\frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right)$
Kaiser	$\frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2-\left(n-\frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$ Where I_o is zero order Bessel function.





Name of window	**Window function.** $\omega(n),\ \lvert n\rvert \le \left(\frac{M-1}{2}\right)$
Hamming	$0.54\text{-}0.46\cos\frac{2\pi n}{M-1}$
Hanning	$\frac{1}{2}\left(1+\cos\frac{2\pi n}{M-1}\right)$

Gibbs phenomenon in FIR filter design:

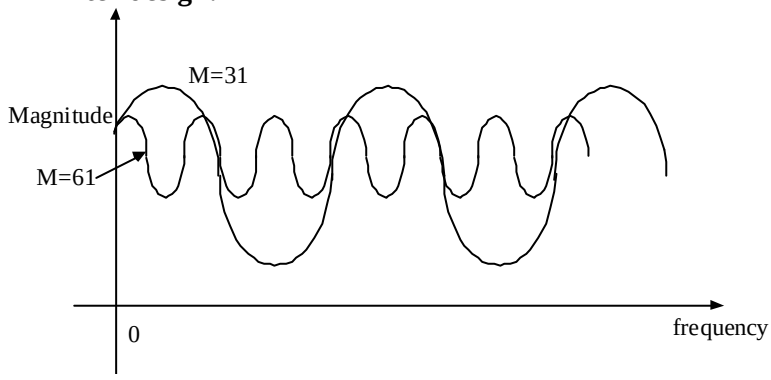


Fig: frequency response of filter.

We observe that relatively large oscillation or ripple occur near the band edge of filter the oscillation increases in the frequency as N increases but they do not diminish in amplitude. Large oscillation are direct result of large side lobes existing in frequency characteristics of $W(\omega)$ of rectangular window. Fourier series representation of $H_d(\omega)$, multiplication of $h_d(\omega)$ with rectangular window is identical to trunked in fourier series representation of desire filter characteristic $H_d(\omega)$. The truncation of fourier series is known to introduce in frequency response characteristic $H(\omega)$ due to non uniform convergence of fourier series at discontinuity. The oscillatory behavior near the band edge of the filer is called Gibbs phenomenon.

To alleviate the presence of large oscillation in both the pass band and stop band we should use a window function that contains a taper and decay towards zero gradually instead of abruptly.

Design of FIR filter by Frequency sampling method:-

In the frequency sampling method of IIR filter design we specified the desire frequency response $H_d(\omega)$ at a set of equally spaced frequency.

Namely $\omega_k = \frac{2\pi}{M}(k+\alpha)$ ……. (1)

Where, k = 0, 1, …….. $\frac{M-1}{2}$ for M odd.

K = 0, 1 ……. $\frac{M}{2}-1$ for M even.

$\alpha = 0, or \frac{1}{2}$

And solve the unit sample respose n/w of the FIR filter from these equally spaced frequency specification. Now the desire frequency response is

$$H(w) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

Suppose that we specify the frequency response of the filter at the frequency given by equation (i).

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

$$H(k+\alpha) = \sum h(n)e^{-j2\pi(k+\alpha)n/M} \quad \text{…… (2)}$$

Where, k = 0, 1, ……. M-1

It is simple matter to invert (2) and express $h(\omega)$ in terms of $H(k+\alpha)$

$$h(n) = \frac{1}{M}\sum_{k=o}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} \quad \text{…… (3)}$$

Where, n = 0, 1 …… M-1

This relationship in (3) allows us to compute the values of unit sample response h(n) from the specification of frequency sample $H(k+\alpha)$, k = 0, 1, …..M-1. Note that when $\alpha = 0$ (2) reduces to discrete fourier transform (DFT) of the sequence h(n) and the expression (3) reduces to IDFT.

Design a low pass FIR filter with 11 coefficient for the following specification.

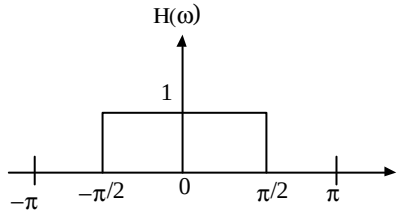
Passband frequency = 0.25 khz

Sampling frequency = 1 khz.

Use rectangular window, Hamming window and hanning window.

Solution:

$f_c = 0.25khz$ $\omega_c = \frac{2\pi \times 0.5}{1} = \frac{\pi}{2}$



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1.e^{j\omega n} d\omega$$
$$= \frac{1}{2}\frac{\sin(n\pi/2)}{(n\pi/2)}$$

$h_d(0) = ½$
$h_d(1) = h_d(-1) = 0.3148$
$h_d(2) = h_d(-2) = 0$
$h_d(3) = h_d(-3) = -0.0162$
$h_d(4) = h_d(-4) = 0$
$h_d(5) = h_d(-5) = 0.06369$

Rectangular window:-
$w(n) = 1 \qquad -5 \le n \le 5$
$h(n) = h_d(n)\, w(n)$
$h(0) = h_d(0)\, w(n) = 0.5$
$h(1) = h_d(-1) = 0.3184$
$h(2) = h_d(-2) = 0$
$h(3) = h_d(-3) = -0.1062$
$h(4) = h_d(-4) = 0$
$h(5) = h_d(-5) = 0.06369$

Hamming window:

$$W_{ham}(n) = 0.54+0.46\cos\left(\frac{2\pi n}{M-1}\right) \qquad -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right)$$
$$= 0.54+0.46\cos\left(\frac{2\pi n}{10}\right) \qquad -5 \le \pi \le 5$$

$h(n) = h_d(n) W_{ham}(n)$
$W_{ham}(0) = 1$
$W_{ham}(1) = W_{ham}(-1) = 0.9121$
$W_{ham}(2) = W_{ham}(-2) = 0.6828$
$W_{ham}(3) = W_{ham}(-3) = 0.3970$
$W_{ham}(4) = W_{ham}(-4) = 0.1679$
$W_{ham}(5) = W_{ham}(-5) = 0.08$

$h(0) = h_d(0)W_{ham}(0) = 0.5$
$h(1) = h(-1) = 0.2904$
$h(2) = h(-2) = 0$
$h(3) = h(-3) = -0.04225$
$h(4) = h(-4) = 0$
$h(5) = h(-5) = 0.005095$

H(z) ……..

Similarly Hanning window……

Q. A low pass filter is required to be design with design frequency response which is expressed as follows, as we

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega} & -\pi/2 \le |\omega| \le \pi/4 \\ 0 & for\ \pi/4 \le |\omega| \le \pi \end{cases}$$

Obtain the filter coefficient $h_d(n)$ if the window function is defined as,

$$W(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & otherwise \end{cases}$$

Q. Design a low pass filter having desire frequency response given as,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & 0 \le \omega \le \pi/2 \\ 0 & \pi/ \le \omega \le \pi \end{cases}$$

Obtain filter coefficient h(n) for M = 7 using frequency sampling method.
Solution:

$$W_k = \frac{2\pi}{M}(k+\alpha)$$
$$\alpha = 0$$
$$h(n) = \frac{1}{7}\sum_{k=0}^{6} e^{-j32\pi k/7} e^{j2\pi kn/7} \qquad n = 0, 1, \ldots\ldots 6.$$
$$h(0) = \frac{1}{7}\sum_{k=0}^{6} e^{-j6\pi k/7} =$$

Remez exchange algorithm (prokish book बाट आफै पढ्ने)

Reference for chapter (8)

- TMS 320 Texas instrument (Sanjaya sharma)
- Bit serial arithmetic (Digital logic book)
- Serial adder.
- Distributed arithmetic and pipelined implementation.